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GM(n,h)模型建模序列数据数乘变换特性研究

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摘 要:基于数乘变换是灰色系统建模过程中数据处理的基础,讨论了 GM(n, h)模型与其他几类灰色模型的内在联系. 将各类灰色模型统一于共同的分析体系,并在此基础上研究了数乘变换对 GM(n, h)模型参数取值的影响. 指出了模型的模拟和预测值只与因变量的数乘变换有关,而与自变量的变换无关. 最后分析了几个特殊灰色模型的数乘变换性质,该结果对研究系列灰色模型参数特征有重要意义.

关键词: GM(n, h)模型; 数乘变换; 最小二乘法; 累加生成
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Research on property of GM(n, h) model under data multiple transformation

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Abstract: Multiple transformation is the basis of data processing to construct a grey model. Connections between the GM(n, h) model and other grey models are discussed. Then several grey models are nuified into a common analysis system. Based on this conclusion, the affections caused by multiple transformation to GM(n, h) model is researched. The values of simulative and predicative are only dependent on the multiple transformation to the main variable and independent to other variables. Finally, the multiple transformation properties of several grey models are analyzed. The results are benefit to research parameters properties of grey models.

Key words: GM(n, h) model; Multiple transformation; Least square methods; Accumulating generator operator

1 引 言

灰色系统建立模型的首要任务是计算模型的参数值,从而建立相应的灰色模型. 目前在灰色系统建模过程中,一般根据收集的原始数据的量级情况进行相应的数据处理,使其消除量纲且具有可比性. 目前常用的变换方法有:初值化变换、均值化变换、归一化变换、区间值化变换以及量级变换[1],这些变换通常都可以归结为数乘变换. 在多变量高阶灰色系统模型中,这些模型数据变换对模型的参数和模拟预测值有何影响? 参数之间的关系怎样? 各种模型在参数变换中的内在联系如何? 这些都是理论上值得探讨的问题.

目前受到关注的灰色系统模型除 GM(1,1)模型外,还有 GM(1, N)模型, GM(0, N)模型和 GM(2,1)模型[2-7]. 关于模型参数特征方面的研究,冯正元[8]研究了灰色直接模型的性质,得到了该模型的参数值与数乘变换序列模型参数值间的量化关系. Li[9]讨论了 GM(1,1)模型的参数值与数乘变换序列模型参数值间的量化关系. 肖新平等[10]研究了 GM(0, N)模型在数乘变换下的参数特征. 对于多变量高阶灰色模型的参数数乘特性的研究和各类灰色模型之间的内在联系也鲜有研究.

本文主要讨论 GM(n, h)模型的参数特性及其与 GM(1,1)模型, GM(1, N)模型, GM(0, N)模型, GM(2,1)模型等模型的内在联系. 通过本研究,可以构建各灰色模型之间的理论桥梁,同时理解灰色模型参数特性的共性特征,对灰色模型系统性研究有着重要意义.

2 对 GM(n, h)模型的理解

GM(n, h)是多阶多变量灰色模型的符号,包括

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n 阶灰导数 h 个因子变量,其符号内涵如图 1 所示.

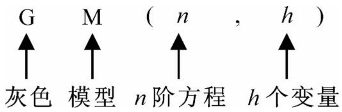

图 1　模型符号内涵

对于 GM(n,h) 模型,其连续形式为

$$\frac{d^n x_1^{(1)}}{dt^n} + a_1 \frac{d^{n-1} x_1^{(1)}}{dt^{n-1}} + a_2 \frac{d^{n-2} x_1^{(1)}}{dt^{n-2}} + \cdots + a_{n-1} \frac{d x_1^{(1)}}{dt} + a_n x_1^{(1)} = b_1 x_2^{(1)} + b_2 x_3^{(1)} + \cdots + b_{h-1} x_h^{(1)} + b_h. \quad (1)$$

由于灰色模型的建模数据不是连续形式的,无法用连续模型来刻画,通常是用差分变换来代替微分变换,即

$$\frac{d^r x_1^{(1)}}{dt^r} = \alpha^{(r)} x_1^{(1)}(k),\ r = 1,2,\cdots,n. \quad (2)$$

通过变换可得模型的定义形式为

$$\alpha^{(n)} x_1^{(1)}(k) + \sum_{i=1}^{n-1} a_i \alpha^{(n-i)} x_1^{(1)}(k) + a_n z_1^{(1)}(k) = \sum_{j=1}^{h-1} b_j x_{j+1}^{(1)}(k) + b_h. \quad (3)$$

其中

$$x_i^{(1)}(k) = \sum_{j=1}^{k} x_i^{(0)}(j),\ i = 1,2,\cdots,h;$$

$$z_1^{(1)}(k) = \frac{1}{2}(x_1^{(1)}(k) + x_1^{(1)}(k-1));$$

$$\alpha^{(r)} x_1^{(1)}(k) = \alpha^{(r-1)} x_1^{(1)}(k) - \alpha^{(r-1)} x_1^{(1)}(k-1),\ r = 1,2,\cdots,n.$$

称模型(3) 为 n 阶 h 个变量灰色模型,记为 GM(n,h).

1) 当 $n = 1, h = 1$ 时,模型(3) 变为

$$\alpha^{(1)} x_1^{(1)}(k) + a_1 z_1^{(1)}(k) = b_1, \quad (4)$$

即为 GM(1,1) 模型.

2) 当 $n = 1, h = N$ 时,模型(3) 变为

$$\alpha^{(1)} x_1^{(1)}(k) + a_1 z_1^{(1)}(k) = b_1 x_2^{(1)}(k) + b_2 x_3^{(1)}(k) + \cdots + b_{N-1} x_N^{(1)}(k) + b_N, \quad (5)$$

即为 GM(1,N) 模型.

3) 当 $n = 0, h = N$ 时,模型(3) 变为

$$x_1^{(1)}(k) = b_1 x_2^{(1)}(k) + b_2 x_3^{(1)}(k) + \cdots + b_{N-1} x_N^{(1)}(k) + b_N, \quad (6)$$

即为 GM(0,N) 模型.

4) 当 $n = 2, h = 1$ 时,模型(3) 变为

$$\alpha^{(2)} x_1^{(1)}(k) + a_1 \alpha^{(1)} x_1^{(1)}(k) + a_2 z_1^{(1)}(k) = b_1, \quad (7)$$

即为 GM(2,1) 模型.

显然,GM(1,1) 模型,GM(1,N) 模型,GM(0,N) 模型和 GM(2,1) 模型等都是由 GM(n,h) 模型简化而来,因此研究 GM(n,h) 模型对灰色模型的研究具有普遍意义,研究 GM(n,h) 模型的参数性质可导出其他模型的参数性质.

3　参数数乘变换特性研究

定义 1　对于 GM(n,h) 模型,定义形式

$$\alpha^{(n)} x_1^{(1)}(k) + \sum_{i=1}^{n-1} a_i \alpha^{(n-i)} x_1^{(1)}(k) + a_n z_1^{(1)}(k) = \sum_{j=1}^{h-1} b_j x_{j+1}^{(1)}(k) + b_h, \quad (8)$$

称 $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_h$ 为 GM(n,h) 模型的参数,记为向量 P_I,即有

$$P_I = [a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_h]^T. \quad (9)$$

命题 1　GM(n,h) 模型参数向量在最小二乘准则下有矩阵算式

$$P_I = [a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_h]^T = (B^T B)^{-1} B^T Y. \quad (10)$$

其中

$$B = \begin{bmatrix} -\alpha^{(n-1)} x_1^{(1)}(n+1) & -\alpha^{(n-2)} x_1^{(1)}(n+1) & \cdots & -\alpha^{(1)} x_1^{(1)}(n+1) & -z_1^{(1)}(n+1) & x_2^{(1)}(n+1) & x_3^{(1)}(n+1) & \cdots & x_h^{(1)}(n+1) & 1 \\ -\alpha^{(n-1)} x_1^{(1)}(n+2) & -\alpha^{(n-2)} x_1^{(1)}(n+2) & \cdots & -\alpha^{(1)} x_1^{(1)}(n+2) & -z_1^{(1)}(n+2) & x_2^{(1)}(n+2) & x_3^{(1)}(n+2) & \cdots & x_h^{(1)}(n+2) & 1 \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ -\alpha^{(n-1)} x_1^{(1)}(m) & -\alpha^{(n-2)} x_1^{(1)}(m) & \cdots & -\alpha^{(1)} x_1^{(1)}(m) & -z_1^{(1)}(m) & x_2^{(1)}(m) & x_3^{(1)}(m) & \cdots & x_h^{(1)}(m) & 1 \end{bmatrix},$$

$$Y = [\alpha^{(n)} x_1^{(1)}(n+1), \alpha^{(n)} x_1^{(1)}(n+2), \cdots, \alpha^{(n)} x_1^{(1)}(m)]^T,$$

m 为数据序列的长度.

证明　从 GM(n,h) 模型定义形式出发,将 $X_i^{(1)} = (x_i^{(1)}(n+1), x_i^{(1)}(n+2), \cdots, x_i^{(1)}(m)), i = 1,2,\cdots,h$ 代入模型,可得矩阵方程 $Y = BP_I$. 考虑矩阵 B 为列满秩的情况,则 $B^T Y = B^T B P_I$. $B^T B$ 可逆,两边同乘 $(B^T B)^{-1}$,可得 $P_I = (B^T B)^{-1} B^T Y$. □

命题 2　令矩阵 T 为 GM(n,h) 模型的中间参数,且

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1,n+h} \\ t_{21} & t_{22} & \cdots & t_{2,n+h} \\ \cdots & \cdots & \ddots & \cdots \\ t_{n+h,1} & t_{n+h,2} & \cdots & t_{n+h,n+h} \end{bmatrix} = (B^T B)^{-1}. \quad (11)$$

命题 3 令

$$C(j) = -\sum_{i=1}^{n-1} t_{j,i} \sum_{k=n+1}^{m} \alpha^{(n-i)} x_1^{(1)}(k)\, \alpha^{(n)} x_1^{(1)}(k),$$

$$D(j) = -t_{j,n} \sum_{k=n+1}^{m} z^{(1)}(k)\, \alpha^{(n)} x_1^{(1)}(k),$$

$$E(j) = \sum_{i=n+1}^{n+h-1} t_{j,i} \sum_{k=n+1}^{m} x_{i-n+1}^{(1)}(k)\, \alpha^{(n)} x_1^{(1)}(k)$$

$$F(j) = t_{j,n+h} \sum_{k=n+1}^{m} \alpha^{(n)} x_1^{(1)}(k),$$

$$j = 1,2,\cdots,n+h$$

为 GM(n,h) 模型的中间参数,则有

$$P_I = [a_1 \quad a_2 \quad \cdots \quad a_n \quad b_1 \quad b_2 \quad \cdots \quad b_h]^{\mathrm{T}} = \begin{bmatrix} C(1) + D(1) + E(1) + F(1) \\ C(2) + D(2) + E(2) + F(2) \\ \cdots \\ C(n+h) + D(n+h) + E(n+h) + F(n+h) \end{bmatrix}. \tag{12}$$

证明 由于 $P_I = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}Y$, 而 $T = (B^{\mathrm{T}}B)^{-1}$, 则 $P_I = TB^{\mathrm{T}}Y$. 写成矩阵形式为

$$P_I = [a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_h]^{\mathrm{T}} = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1,n+h} \\ t_{21} & t_{22} & \cdots & t_{2,n+h} \\ \cdots & \cdots & \ddots & \cdots \\ t_{n+h,1} & t_{n+h,2} & \cdots & t_{n+h,n+h} \end{bmatrix} \times$$

$$\begin{bmatrix} -\alpha^{(n-1)} x_1^{(1)}(n+1) & -\alpha^{(n-1)} x_1^{(1)}(n+2) & \cdots & -\alpha^{(n-1)} x_1^{(1)}(m) \\ -\alpha^{(n-2)} x_1^{(1)}(n+1) & -\alpha^{(n-2)} x_1^{(1)}(n+2) & \cdots & -\alpha^{(n-2)} x_1^{(1)}(m) \\ \cdots & \cdots & \ddots & \cdots \\ -\alpha^{(1)} x_1^{(1)}(n+1) & -\alpha^{(1)} x_1^{(1)}(n+2) & \cdots & -\alpha^{(1)} x_1^{(1)}(m) \\ -z^{(n)}(n+1) & -z^{(1)}(n+2) & \cdots & -z^{(1)}(m) \\ x_2^{(1)}(n+1) & x_2^{(1)}(n+2) & \cdots & x_2^{(1)}(m) \\ x_3^{(1)}(n+1) & x_3^{(1)}(n+2) & \cdots & x_3^{(1)}(m) \\ \cdots & \cdots & \ddots & \cdots \\ x_h^{(1)}(n+1) & x_h^{(1)}(n+2) & \cdots & x_h^{(1)}(m) \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \alpha^{(n)} x_1^{(1)}(n+1) \\ \alpha^{(n)} x_1^{(1)}(n+2) \\ \cdots \\ \alpha^{(n)} x_1^{(1)}(m) \end{bmatrix}.$$

展开可得

$$a_p = -\sum_{i=1}^{n-1} t_{p,i} \sum_{k=n+1}^{m} \alpha^{(n-i)} x_1^{(1)}(k)\, \alpha^{(n)} x_1^{(1)}(k) - t_{p,n} \sum_{k=n+1}^{m} z^{(1)}(k)\, \alpha^{(n)} x_1^{(1)}(k) + \sum_{t=n+1}^{n+h-1} t_{p,i} \sum_{k=n+1}^{m} x_{i-n+1}^{(1)}(k)\, \alpha^{(n)} x_1^{(1)}(k) + t_{p,n+h} \sum_{k=n+1}^{m} \alpha^{(n)} x_1^{(1)}(k),$$

$$b_s = -\sum_{i=1}^{n-1} t_{n+s,i} \sum_{k=n+1}^{m} \alpha^{(n-i)} x_1^{(1)}(k)\, \alpha^{(n)} x_1^{(1)}(k) - t_{n+s,n} \sum_{k=n+1}^{m} z^{(1)}(k)\, \alpha^{(n)} x_1^{(1)}(k) + \sum_{i=n+1}^{n+h-1} t_{n+s,i} \sum_{k=n+1}^{m} x_{i-n+1}^{(1)}(k)\, \alpha^{(n)} x_1^{(1)}(k) + t_{n+s,n+h} \sum_{k=n+1}^{m} \alpha^{(n)} x_1^{(1)}(k),$$

$$p = 1,2,\cdots,n;\ s = 1,2,\cdots,h.$$

令

$$C(j) = -\sum_{i=1}^{n-1} t_{j,i} \sum_{k=n+1}^{m} \alpha^{(n-i)} x_1^{(1)}(k)\, \alpha^{(n)} x_1^{(1)}(k),$$

$$D(j) = -t_{j,n} \sum_{k=n+1}^{m} z^{(1)}(k)\, \alpha^{(n)} x_1^{(1)}(k),$$

$$E(j) = \sum_{i=n+1}^{n+h-1} t_{j,i} \sum_{k=n+1}^{m} x_{i-n+1}^{(1)}(k)\, \alpha^{(n)} x_1^{(1)}(k),$$

$$F(j) = t_{j,n+h} \sum_{k=n+1}^{m} \alpha^{(n)} x_1^{(1)}(k),$$

$$j = 1,2,\cdots,n+h.$$

显然可得式(12). □

下面讨论数乘变换对 GM(n,h) 模型的影响. 设 $Y_i^{(0)}$ 为第 i 个原始数据序列, $X_i^{(0)}$ 为数乘变换序列, 有

$$Y_i^{(0)} = (y_i^{(0)}(1), y_i^{(0)}(2), \cdots, y_i^{(0)}(m)),$$
$$X_i^{(0)} = (x_i^{(0)}(1), x_i^{(0)}(2), \cdots, x_i^{(0)}(m)),$$
$$x_i^{(0)}(k) = \sigma_i y_i^{(0)}(k),$$
$$i = 1,2,\cdots,h,\ k = 1,2,\cdots,m. \tag{13}$$

原始数据序列 $Y_i^{(0)}$ 的 GM(n,h) 模型参数和数据矩阵分别为 $a_{y1}, a_{y2}, \cdots, a_{yn}, b_{y1}, b_{y2}, \cdots, b_{yh}$ 和 B_y, 数乘序列 $X_i^{(0)}$ 的 GM(n,h) 模型的对应参数和数据矩阵分别为 $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_h$ 和 B, 其他参数类似定义.

定理 1 设 B 为 GM(n,h) 模型的数据矩阵, 记

$$B = (B_{ij})_{(m-n)\times(n+h)}. \tag{14}$$

其中

$$B_{ij} = -\alpha^{(n-j)} x_1^{(1)}(n+i),$$
$$i = 1,2,\cdots,m-n,\ j = 1,2,\cdots,n-1;$$
$$B_{in} = -z_1^{(1)}(n+i),\ i = 1,2,\cdots,m-n;$$

$$B_{ij} = x^{(1)}_{j-n+1}(n+i),\ i = 1,2,\cdots,m-n,$$
$$j = n+1,n+2,\cdots,n+h-1;$$
$$B_{i,n+h} = 1,\ i = 1,2,\cdots,m-n.$$

则

$$B_{ij} = \sigma_1 B_{yij},$$
$$i = 1,2,\cdots,m-n,\ j = 1,2,\cdots,n-1;$$
$$B_{in} = \sigma_1 B_{yin},\ i = 1,2,\cdots,m-n;$$
$$B_{ij} = \sigma_{j-n+1} B_{yij},\ i = 1,2,\cdots,m-n,$$
$$j = n+1,n+2,\cdots,n+h-1;$$
$$B_{i,n+h} = B_{yi,n+h},\ i = 1,2,\cdots,m-n. \quad (15)$$

证明　因为

$$x_i^{(1)}(k) = \sum_{j=1}^{k} x_i^{(0)}(j) = \sum_{j=1}^{k} \sigma_i y_i^{(0)}(j) =$$
$$\sigma_i \sum_{j=1}^{k} y_i^{(0)}(j) = \sigma_i y_i^{(0)}(k),\ i = 1,2,\cdots,h;$$
$$z_1^{(1)}(k) = \frac{1}{2}(x_1^{(1)}(k) + x_1^{(1)}(k-1)) =$$
$$\frac{1}{2}(\sigma_1 y_1^{(1)}(k) + \sigma_1 y_1^{(1)}(k-1)) = \sigma_1 z_{y1}^{(1)}(k);$$
$$\alpha^{(1)} x_1^{(1)}(k) = x_1^{(1)}(k) - x_1^{(1)}(k-1) =$$
$$\sigma_1 y_1^{(1)}(k) - \sigma_1 y_1^{(1)}(k-1) = \sigma_1 \alpha^{(1)} y_1^{(1)}(k);$$
$$\alpha^{(r)} x_1^{(1)}(k) = x_1^{(r-1)}(k) - x_1^{(r-1)}(k-1) =$$
$$\sigma_1 y_1^{(r-1)}(k) - \sigma_1 y_1^{(r-1)}(k-1) = \sigma_1 \alpha^{(1)} y_1^{(r)}(k);$$

所以

$$B_{ij} = -\alpha^{(n-j)} x_1^{(1)}(n+i) =$$
$$-\sigma_1 \alpha^{(n-j)} y_1^{(1)}(n+i) = \sigma_1 B_{yij},$$
$$j = 1,2,\cdots,n-1;$$
$$B_{in} = -z_1^{(1)}(n+i) = -\sigma_1 z_{y1}^{(1)}(n+i) = \sigma_1 B_{yin},$$
$$B_{ij} = x^{(1)}_{j-n+1}(n+i) =$$
$$\sigma_{j-n+1} y^{(1)}_{j-n+1}(n+i) = \sigma_{j-n+1} B_{yij},$$
$$j = n+1,n+2,\cdots,n+h-1;$$
$$B_{i,n+h} = 1 = B_{yi,n+h}.$$
□

为以后讨论方便,令

$$\rho_1 = \rho_2 = \cdots = \rho_n = \sigma_1,$$
$$\rho_{n+i-1} = \sigma_i,\ i = 2,3,\cdots,h,\ \rho_{n+h} = 1,$$

则可将 B 中的元素改写为

$$B_{ij} = \rho_j B_{yi,j},$$
$$i = 1,2,\cdots,m-n,\ j = 1,2,\cdots,n+h,$$

即

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1,n+h} \\ B_{21} & B_{22} & \cdots & B_{2,n+h} \\ \cdots & \cdots & \ddots & \cdots \\ B_{m-n,1} & B_{m-n,2} & \cdots & B_{m-n,n+h} \end{bmatrix} =$$
$$\begin{bmatrix} \rho_1 B_{y11} & \rho_2 B_{y12} & \cdots & \rho_{n+h} B_{y1,n+h} \\ \rho_1 B_{y21} & \rho_2 B_{y22} & \cdots & \rho_{n+h} B_{y2,n+h} \\ \cdots & \cdots & \ddots & \cdots \\ \rho_1 B_{y(m-n),1} & \rho_2 B_{y(m+n),2} & \cdots & \rho_{n+h} B_{y(m-n),n+h} \end{bmatrix}, \quad (16)$$

则有 $B_{ij}^{T} = \rho_i B_{yi,j}^{T}$.

定理 2　数据矩阵 B 如定理 1 所述,记

$$B^{T}B = D = (d_{ij})_{(n+h)\times(n+h)}. \quad (17)$$

其中: $d_{ij} = \sum_{k=1}^{m-n} B_{ik}^{T} B_{kj}$, $i,j = 1,2,\cdots,n+h$; D^{*} 为 D 的伴随矩阵且 $D^{*} = (D_{ij})_{(n+h)\times(n+h)}$, D_{ij} 是 D 中元素 d_{ij} 的代数余子式,有:

1) $d_{ij} = \rho_i\rho_j d_{yi,j}$, $i,j = 1,2,\cdots,n+h$;

2) D 和 D^{*} 为对称矩阵.

证明　1)

$$d_{ij} = \sum_{k=1}^{m-n} B_{ik}^{T} B_{kj} = \sum_{k=1}^{m-n} \rho_i B_{yik}^{T} \rho_j B_{ykj} =$$
$$\rho_i\rho_j \sum_{k=1}^{m-n} B_{yi,k}^{T} B_{yk,j} = \rho_i\rho_j d_{yi,j},\ i,j = 1,2,\cdots,n+h.$$

2) 因为

$$D^{T} = (B^{T}B)^{T} = B^{T}(B^{T})^{T} = B^{T}B = D,$$
$$(D^{*})^{T} = (D^{T})^{*} = D^{*},$$

所以 D 和 D^{*} 为对称矩阵. □

定理 3　设向量 $[a_{y1}, a_{y2}, \cdots, a_{yn}, b_{y1}, b_{y2}, \cdots, b_{yh}]^{T}$ 和 $[a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_h]^{T}$ 分别是建立在原始数据序列 $Y_i^{(0)}$ 和数乘序列 $X_i^{(0)}$ 上的 GM(n,h) 模型的参数向量, $x_i^{(0)}(k) = \sigma_i y_i^{(0)}(k)$, $i = 1,2,\cdots,h$, $k = 1,2,\cdots,m$,则有

$$a_i = a_{yi},\ i = 1,2,\cdots,n;$$
$$b_j = \frac{\sigma_1}{\sigma_{j+1}} b_{yj},\ j = 1,2,\cdots,h-1;$$
$$b_h = \sigma_1 b_{yh}. \quad (18)$$

证明　记

$$\rho_1 = \rho_2 = \cdots = \rho_n = \sigma_1,\ \rho_{n+h} = 1,$$
$$\rho_{n+i-1} = \sigma_i,\ i = 2,3,\cdots,h,$$
$$D^{-1}B^{T} = E = (e_{ik})_{(n+h)\times(m-n)},$$

且设

$$\rho(i,j) = \rho_1^2 \cdots \rho_{i-1}^2 \rho_i \rho_{i+1}^2 \cdots \rho_{j-1}^2 \rho_j \rho_{j+1}^2 \cdots \rho_{n+h}^2,$$
$$\rho(i) = \rho_1^2 \cdots \rho_{i-1}^2 \rho_i \rho_{i+1}^2 \cdots \rho_{n+h}^2,$$
$$\rho(0) = \rho_1^2 \rho_2^2 \cdots \rho_{n+h}^2.$$

先证明 $D_{ij} = \rho(i,j) D_{yij}$, $|D| = \rho(0)|D_y|$,其中 $|D|$ 和 $|D_y|$ 分别为矩阵 D 和 D_y 的行列式. 由代数余子式定义有

$$D_{ij} =$$
$$(-1)^{i+j} \times$$

$$\begin{vmatrix} d_{11} & d_{12} & \cdots & d_{1,j-1} & d_{1,j+1} & \cdots & d_{1,n+h} \\ d_{21} & d_{22} & \cdots & d_{2,j-1} & d_{2,j+1} & \cdots & d_{2,n+h} \\ \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots \\ d_{i-1,1} & d_{i-1,2} & \cdots & d_{i-1,j-1} & d_{i-1,j+1} & \cdots & d_{i-1,n+h} \\ d_{i+1,1} & d_{i+1,2} & \cdots & d_{i+1,j-1} & d_{i+1,j+1} & \cdots & d_{i+1,n+h} \\ \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots \\ d_{n+h,1} & d_{n+h,2} & \cdots & d_{n+h,j-1} & d_{n+h,j+1} & \cdots & d_{n+h,n+h} \end{vmatrix}.$$

根据定理 2 有 $d_{ij} = \rho_i\rho_j d_{yij}$，因此可得

$$D_{ij} = (-1)^{i+j} \times$$

$$\begin{vmatrix} \rho_1^2 d_{y11} & \rho_1\rho_2 d_{y12} & \cdots & \rho_1\rho_{j-1} d_{y1,j-1} & \rho_1\rho_{j+1} d_{y1,j+1} & \cdots & \rho_1\rho_{n+h} d_{y1,n+h} \\ \rho_2\rho_1 d_{y21} & \rho_2^2 d_{y22} & \cdots & \rho_2\rho_{j-1} d_{y2,j-1} & \rho_2\rho_{j+1} d_{y2,j+1} & \cdots & \rho_2\rho_{n+h} d_{y2,n+h} \\ \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots \\ \rho_{i-1}\rho_1 d_{y(i-1),1} & \rho_{i-1}\rho_2 d_{y(i-1),2} & \cdots & \rho_{i-1}\rho_{j-1} d_{y(i-1),j-1} & \rho_{i-1}\rho_{j+1} d_{y(i-1),j+1} & \cdots & \rho_{i-1}\rho_{n+h} d_{y(i-1),n+h} \\ \rho_{i+1}\rho_1 d_{y(i+1),1} & \rho_{i+1}\rho_2 d_{y(i+1),2} & \cdots & \rho_{i+1}\rho_{j-1} d_{y(i+1),j-1} & \rho_{i+1}\rho_{j+1} d_{y(i+1),j+1} & \cdots & \rho_{i+1}\rho_{n+h} d_{y(i+1),n+h} \\ \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots \\ \rho_{n+h}\rho_1 d_{y(n+h),1} & \rho_{n+h}\rho_2 d_{y(n+h),2} & \cdots & \rho_{n+h}\rho_{j-1} d_{y(n+h),j-1} & \rho_{n+h}\rho_{j+1} d_{y(n+h),j+1} & \cdots & \rho_{n+h}^2 d_{y(n+h),n+h} \end{vmatrix} =$$

$$(-1)^{i+j}\rho_1^2\cdots\rho_{i-1}^2\rho_i\rho_{i+1}^2\cdots\rho_{j-1}^2\rho_j\rho_{j+1}^2\cdots\rho_{n+h}^2 \times$$

$$\begin{vmatrix} d_{y11} & d_{y12} & \cdots & d_{y1,j-1} & d_{y1,j+1} & \cdots & d_{y1,n+h} \\ d_{y21} & d_{y22} & \cdots & d_{y2,j-1} & d_{y2,j+1} & \cdots & d_{y2,n+h} \\ \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots \\ d_{y(i-1),1} & d_{y(i-1),2} & \cdots & d_{y(i-1),j-1} & d_{y(i-1),j+1} & \cdots & d_{y(i-1),n+h} \\ d_{y(i+1),1} & d_{y(i+1),2} & \cdots & d_{y(i+1),j-1} & d_{y(i+1),j+1} & \cdots & d_{y(i+1),n+h} \\ \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots \\ d_{y(n+h),1} & d_{y(n+h),2} & \cdots & d_{y(n+h),j-1} & d_{y(n+h),j+1} & \cdots & d_{y(n+h),n+h} \end{vmatrix} = \rho(i,j) D_{yij}.$$

将矩阵 D 的行列式元素用 $d_{ij} = \rho_i\rho_j d_{yi,j}$ 替换，得

$$|D| = \begin{vmatrix} d_{11} & d_{12} & \cdots & d_{1,n+h} \\ d_{21} & d_{22} & \cdots & d_{2,n+h} \\ \cdots & \cdots & \ddots & \cdots \\ d_{n+h,1} & d_{n+h,2} & \cdots & d_{n+h,n+h} \end{vmatrix} =$$

$$\begin{vmatrix} \rho_1^2 d_{y1,1} & \rho_1\rho_2 d_{y1,2} & \cdots & \rho_1\rho_{n+h} d_{y1,n+h} \\ \rho_2\rho_1 d_{y2,1} & \rho_2^2 d_{y2,2} & \cdots & \rho_2\rho_{n+h} d_{y2,n+h} \\ \cdots & \cdots & \ddots & \cdots \\ \rho_{n+h}\rho_1 d_{y(n+h),1} & \rho_{n+h}\rho_2 d_{y(n+h),2} & \cdots & \rho_{n+h}^2 d_{y(n+h),n+h} \end{vmatrix} =$$

$$\rho(0)\ |D_y|.$$

根据 e_{ik} 的定义，得到

$$e_{ik} = \frac{1}{|D|}\sum_{j=1}^{n+h} D_{ij} B_{jk}^{\mathrm{T}} = \frac{1}{\rho(0)\ |D_y|}\sum_{j=1}^{n+h}\rho(i,j) D_{yi,j}\rho_j B_{yj,k}^{\mathrm{T}} \overset{\rho(i)=\rho(i,j)\rho_j}{=} \frac{1}{\rho(0)\ |D_y|}\sum_{j=1}^{n+h}\rho(i) D_{yi,j} B_{yj,k}^{\mathrm{T}} \overset{1/\rho_i=\rho(i)/\rho(0)}{=} \frac{1}{\rho_i\ |D_y|}\sum_{j=1}^{n+h} D_{yi,j} B_{yj,k}^{\mathrm{T}} = \frac{1}{\rho_i} e_{yi,k}.$$

从而根据命题 3 有

$$a_i = C(i) + D(i) + E(i) + F(i),\quad i = 1,2,\cdots,n.$$

可推导出

$$a_i = \sum_{k=1}^{m-n} e_{ik}\alpha^{(n)} x_1^{(1)}(n+k) = \sum_{k=1}^{m-n}\frac{1}{\rho_i} e_{yik}\sigma_1\alpha^{(n)} y_1^{(1)}(n+k) = \frac{\sigma_1}{\rho_i}\sum_{k=1}^{m-n} e_{yik}\alpha^{(n)} y_1^{(1)}(n+k) = \frac{\sigma_1}{\rho_i} a_{yi} = a_{yi},\quad i = 1,2,\cdots,n.$$

同理可得

$$b_j = \sum_{k=1}^{m-n} e_{j+n,k}\alpha^{(n)} x_1^{(1)}(n+k) = \sum_{k=1}^{m-n}\frac{1}{\rho_{j+n}} e_{y(j+n),k}\sigma_1\alpha^{(n)} y_1^{(1)}(n+k) = \frac{\sigma_1}{\rho_{j+n}}\sum_{k=1}^{m-n} e_{y(j+n),k}\alpha^{(n)} y_1^{(1)}(n+k) = \frac{\sigma_1}{\rho_{j+n}} b_{yj} = \frac{\sigma_1}{\sigma_{j+1}} b_{yj},\quad j = 1,2,\cdots,h.$$

综上，定理 3 得证. □

4　模型数乘变换结果对比分析

根据定理 3 研究的参数 a_i 和 b_j 的数乘变换结果，可分析原始数据序列和数乘序列 GM(n,h) 模型的数据变换结果的对比情况，下面针对 GM(1,1) 模型，GM$(1,N)$ 模型，GM$(0,N)$ 模型和 GM(2,1) 模型进行分析.

推论 1　设 $\hat{x}_1^{(0)}(k)$，$\hat{y}_1^{(0)}(k)$ 分别为序列 X_1，Y_1 的 GM(1,1) 模型模拟值(或预测值)，则模型解之间的关系为

$$\hat{x}_1^{(0)}(k) = \sigma_1 \hat{y}_1^{(0)}(k). \tag{19}$$

证明 由 GM(1,1) 模型的计算公式得

$$\hat{x}_1^{(0)}(k) = \hat{x}_1^{(1)}(k) - \hat{x}_1^{(1)}(k-1) =$$

$$\left(x^{(0)}(1) - \frac{b_1}{a_1}\right)(1 - e^{a_1})e^{-a_1(k-1)} =$$

$$\left(\sigma_1 y^{(0)}(1) - \frac{\sigma_1 b_{y1}}{a_{y1}}\right)(1 - e^{a_{y1}})e^{-a_{y1}(k-1)} =$$

$$\sigma_1\left(y^{(0)}(1) - \frac{b_{y1}}{a_{y1}}\right)(1 - e^{a_{y1}})e^{-a_{y1}(k-1)} =$$

$$\sigma_1 \hat{y}_1^{(0)}(k).$$ □

推论 2 设 $\hat{x}_1^{(0)}(k)$, $\hat{y}_1^{(0)}(k)$ 分别为序列 X_i, Y_i 的 GM(1, N) 模型模拟值(或预测值), 则模型解之间的关系与式(19) 相同.

证明类似于推论 1, 此略.

推论 3 设 $\hat{x}_1^{(0)}(k)$, $\hat{y}_1^{(0)}(k)$ 分别为序列 X_i, Y_i 的 GM(0, N) 模型模拟值(或预测值), 则模型解之间的关系与式(19) 相同.

证明类似于推论 1, 此略.

推论 4 设 $\hat{x}_1^{(0)}(k)$, $\hat{y}_1^{(0)}(k)$ 分别为序列 X_i, Y_i 的 GM(2,1) 模型模拟值(或预测值), 则模型解之间的关系与式(19) 相同.

证明类似于推论 1, 此略.

5 结 论

本文分析了 GM(n, h) 模型与 GM(1,1) 模型, GM(0, N) 模型, GM(1, N) 模型, GM(2,1) 模型的内在联系, 将这几类灰色模型统一到 GM(n, h) 模型分析体系下, 使其具有更广泛的研究意义. 而 GM(1,1) 模型, GM(0, N) 模型, GM(1, N) 模型和 GM(2,1) 模型都是 GM(n, h) 模型的特例, 因此研究 GM(n, h) 模型的参数数乘变换特征即可知道其他模型参数共性的数乘变换特征.

系统的模型值只与系统主行为原始序列 $X_1^{(0)}$ 的数乘变换值 σ_1 有关, 而与其他因子序列的数乘变换值 σ_2, σ_3, …, σ_h 无关, 这在多因素灰色系统模型中有着重要的实际意义. 即无论系统行为因子的量纲是否相同, 在 GM(n, h) 模型建模过程中, 可以根据计算量的大小、数据的实际意义等因素, 不用考虑数据本身的影响, 预先对数据序列做数乘变换, 缩小数据的量级, 简化建模过程, 且不会改变模型的模拟和预测效果.

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