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网络稳定性与控制的小增益原理:回顾与近期进展

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摘 要: 小增益定理是现代控制理论中极为重要的基本工具之一,它在关联系统和不确定系统的鲁棒稳定性分析以及鲁棒控制器设计的许多工作中都发挥着极大的作用. 基于输入到状态稳定性的概念,笔者于1994年首次提出了广义非线性小增益定理. 与之前的小增益定理不同,这一结果为同时刻画关联系统的内部稳定性和外部稳定性提供了一个统一的框架. 从镇定与鲁棒自适应控制到分散式或分布式控制以及输出调节 (抗干扰渐近跟踪),基于非线性小增益定理已经发展出一系列的鲁棒非线性控制器设计新工具. 在过去10年间,复杂非线性大系统已成为研究热点,驱动着小增益定理向更加完备的网络小增益定理方向发展,以期解决网络稳定性与控制中的新问题. 对此,针对非线性小增益理论的一些最新研究进展及其在通讯和计算约束下的网络化控制和事件驱动控制应用结果进行综述,并对该理论的未来研究方向给出一些建议.

关键词: 非线性小增益定理；输入到状态稳定性；关联系统；大规模动态网络

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A small-gain principle for network stability and control: An overview and recent results

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Abstract: The small-gain theorem is one of the fundamentally important and systematic tools in modern control theory. Its power in testing robustness of stability and designing robust controllers for interconnected and uncertain systems has clearly been demonstrated in the work of many others. Taking explicit advantage of input-to-state stability(ISS), the first generalized nonlinear ISS small-gain theorem was proposed by one of the authors in 1994. The nonlinear ISS small-gain theorem distinguishes itself from earlier small-gain theorems by providing a unified framework for internal stability and external stability of interconnected systems. Applications to a variety of control problems ranging from stabilization and robust adaptive control to decentralized or distributed control and output regulation (asymptotic tracking with disturbance rejection) have generated several novel tools for the design of robust nonlinear controllers. In the past ten years, renewed interest in large-scale nonlinear systems has motivated the further development of small-gain theorems toward a complete network small-gain theory for network stability and control. This paper provides a survey of some recent developments of the nonlinear small-gain theory and its applications in networked control systems and event-based control subject to communications and computation constraints, and gives some suggestions on this theory and future research direction.

Keywords: nonlinear small-gain theorem; input-to-state stability; interconnected systems; large-scale dynamical networks

0 Introduction

The small-gain condition, essentially asking the loop-gain to be less than unity, is one way to ensure stability of interconnected systems. The original idea of the small-gain theorem was studied for the negative feedback interconnection of two finite-gain stable

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systems with the gain property taking a linear or affine form; e.g., [1-2] for input-output stability properties of feedback systems; also see the recent works [3-4]. A generalization of the classical small-gain theorem for nonlinear feedback systems with non-affine gains was presented in [5-6], again purely within the input-output context.

By making use of Sontag's seminal work in input-to-state stability (ISS)[7-9], the first generalized, nonlinear ISS small-gain theorem was proposed in [10]. The variants of the small-gain theorem, using input-to-state practical stability (ISpS) and input-to-output stability (IOS), are also available in [10]. As a fundamental difference with respect to the earlier small-gain theorems, in the ISS or IOS framework, the role of the initial conditions is made explicit to ensure internal stability such as asymptotic stability in the Lyapunov sense and external stability such as bounded-input bounded-output (BIBO) stability. A new small-gain tool was presented for the first time in [10-11] for the robust global stabilization of a class of nonlinear systems with dynamic uncertainties. In parallel, Teel[12] presented a small-gain tool for the analysis and synthesis of control systems with input saturation. A Lyapunov reformulation of the ISS small-gain theorem can be found in [13]. Necessary and sufficient small-gain conditions for interconnected integral input-to-state stable (iISS) systems can be found in [14-16]. Further extensions of the small-gain theorem have also been made for general nonlinear systems possibly with time delays in [17] using the concept of vector Lyapunov functions, in [16, 18] using Lyapunov-Krasovskii functionals, and in [19] using the Ruzumikhin technique. As a powerful tool, the ISS small-gain theorem has been included in standard textbooks on nonlinear systems; e.g., [20-21]. See also the book [22] and the references cited therein for other more recent developments along the line of ISS small-gain.

Due to the renewed interest in large-scale nonlinear systems, [23] stated an extension of the nonlinear small-gain theorem for the first time, for networks of discrete-time ISS systems. Shortly, [24-26] developed a matrix-small-gain criterion for networks with plus-type interconnections. In [27-28], a more general cyclic-small-gain theorem for networks of IOS systems was developed. The corresponding Lyapunov formulations have been developed in [29-30]. It should be noted that the matrix-small-gain condition is given by matrix inequalities of nonlinear functions, which is usually not easily checkable. As shown in this paper, the cyclic-small-gain condition can be easily verified by directly testing the loop-gains (i.e. compositions of the ISS or IOS gains of the subsystems along any simple cycle).

The small-gain methods have also been introduced to hybrid systems, which involve both continuous-time and discrete-time dynamics; e.g., [17, 31-37]. In [34], the impulses are time-triggered and a (converse) dwell-time-based strategy was developed to evaluate the ISS property of impulsive systems. In [36], the discrete evolution is state-triggered and both the continuous evolution and the discrete evolution are required to possess some stability property to guarantee the ISS of a hybrid system. ISS small-gain criteria for hybrid feedback systems and their corresponding Lyapunov formulations have also been developed by [31, 33, 35]. The interest in these results for quantized control, impulsive control, and networked control can be found in recent papers; e.g., [32, 34]. In [38], the authors considered nonlinear systems with discontinuous right-hand sides. [17, 37] further generalized the small-gain results to large-scale hybrid dynamic networks, based on vector Lyapunov functions and the matrix-small-gain theorem, respectively. Recent results in robust global stabilization of nonlinear systems based on vector-control Lyapunov functions can be found in [39-40]. It should be pointed out that, in [31, 35, 37], the impulses of the subsystems are supposed to be triggered at the same time. A cyclic-small-gain theorem for hybrid dynamic networks with the impulses of the subsystems triggered asynchronously was developed in [41]. A time-delay version of the cyclic-small-gain theorem can be found in [42].

This paper briefly reviews the development of the nonlinear small-gain theorem and introduces some recent results based on nonlinear small-gain techniques. The rest of the paper is organized as follows. Section 1 reviews the nonlinear small-gain theorem for interconnected systems composed of two subsystems, and Section 2 introduces the extension to large-scale nonlinear systems composed of more than two subsystems. This paper mainly focuses on continuous-time systems represented by differential equations, while the counterparts of the introduced results for discrete-time systems, hybrid systems and delayed systems are also given. Applications to control designs via small-gain techniques are briefly discussed in Section 3. Conclusions and future work are summarized in Section 4.

1 Interconnected nonlinear systems

This section introduces the first, fundamentally nonlinear variant of the classic small-gain theorem, known as the ISS small-gain theorem, as well as its related methods and results. With the ISS small-gain theorem, the ISS property of an interconnected system composed of two ISS subsystems can be tested by checking the composition of the ISS gains of two subsystems. Specifically, Subsection 1.1 gives the trajectory-based small-gain theorem; see [10] for the original reference. Due to the importance of Lyapunov functions, the Lyapunov-based ISS small-gain theorem originally developed in [13] is given in Subsection 1.2. Note that small-gain results for more general systems, e.g., systems modeled by retarded functional differential equations, have also been developed in the literature; e.g., [22].

1.1 Trajectory-based small-gain theorem

Consider a general interconnected nonlinear system composed of two subsystems:

$$\dot{x}_1 = f_1(x, u_1), \tag{1}$$

$$\dot{x}_2 = f_2(x, u_2), \tag{2}$$

where $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ with $x_1 \in \boldsymbol{R}^{n_1}$ and $x_2 \in \boldsymbol{R}^{n_2}$ is the state, $u_1 \in \boldsymbol{R}^{m_1}$ and $u_2 \in \boldsymbol{R}^{m_2}$ are the external inputs, and $f_1 : \boldsymbol{R}^{n_1+n_2} \times \boldsymbol{R}^{m_1} \to \boldsymbol{R}^{n_1}$ and $f_2 : \boldsymbol{R}^{n_1+n_2} \times \boldsymbol{R}^{m_2} \to \boldsymbol{R}^{n_2}$ are locally Lipschitz functions satisfying $f_1(0,0) = 0$ and $f_2(0,0) = 0$. For convenience of notation, define $u = [u_1^{\mathrm{T}}, u_2^{\mathrm{T}}]^{\mathrm{T}}$. By considering u as a function of time, assume that it is measurable and locally essentially bounded. An interconnected system with external inputs is shown in Figure 1.

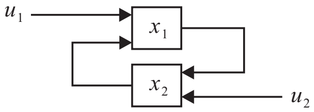

Figure 1　An interconnected system with external inputs

For $i = 1, 2$, assume that each x_i-subsystem is ISS with x_{3-i} and u_i as the inputs. Specifically, for each $i = 1, 2$, there exist $\beta_i \in \mathcal{KL}$ and $\gamma_{i(3-i)}, \gamma_i^u \in \mathcal{K}$ such that for any initial state $x_i(0) = x_{i0}$ and any measurable and locally essentially bounded inputs x_{3-i}, u_i, it holds that

$$|x_i(t)| \leqslant \max\{\beta_i(|x_{i0}|, t), \gamma_{i(3-i)}(\|x_{3-i}\|_\infty), \gamma_i^u(\|u_i\|_\infty)\} \tag{3}$$

for all $t \geqslant 0$. Here, the ISS property of each individual subsystem is expressed using the “max” form, which is mathematically equivalent to the “plus” form used in [10], possibly with different pairs (β, γ).

With the discussions below, we show that the interconnected system is ISS with u as the input if

$$\gamma_{12} \circ \gamma_{21} < \mathrm{Id}. \tag{4}$$

By $\gamma_{12} \circ \gamma_{21} < \mathrm{Id}$, we mean $\gamma_{12}(\gamma_{21}(s)) < s$ for all $s > 0$. It should be noted that for any $\gamma_{12}, \gamma_{21} \in \mathcal{K}$,

$$\gamma_{12} \circ \gamma_{21} < \mathrm{Id} \Leftrightarrow \gamma_{21} \circ \gamma_{12} < \mathrm{Id}. \tag{5}$$

Indeed, for the implication “$\Rightarrow$”, assume that $\gamma_{21} \circ \gamma_{12} < \mathrm{Id}$ does not hold. That is, there exists a positive s such that $\gamma_{21}(\gamma_{12}(s)) \geqslant s$. Then, $\gamma_{12} \circ \gamma_{21}(\gamma_{12}(s)) \geqslant \gamma_{12}(s)$, which leads to a contradiction with $\gamma_{12} \circ \gamma_{21} < \mathrm{Id}$. By symmetry, the other implication “$\Leftarrow$” holds.

Theorem 1 presents a trajectory-based ISS small-gain result.

Theorem 1　Consider the interconnected system composed of two subsystems in the form of (1) and (2) satisfying (3). The interconnected system is ISS with u as the input if the small-gain condition (4) is satisfied.

A special, yet interesting, case of the interconnected system is a cascade system for which one of the gains γ_{12} and γ_{21} is zero. In this case, the small-gain condition is satisfied automatically. If, moreover, $(u_1, u_2) = (0, 0)$, then Theorem 1 recovers Lemma 4.7[21] as a special case for global asymptotic stability (GAS).

The small-gain result developed in [10] can cover the more general case in which the subsystems are interconnected with each other by outputs instead of states. Consider the following interconnected system:

$$\dot{x}_i = f_i(x_i, y_{3-i}, u_i), \tag{6}$$

$$y_i = h_i(x_i), \tag{7}$$

where, for $i = 1, 2, x_i \in \boldsymbol{R}^{n_i}$ is the state, $u_i \in \boldsymbol{R}^{m_i}$ is the input, $y_i \in \boldsymbol{R}^{l_i}$ is the output, and f_i, h_i are locally Lipschitz functions satisfying $f_i(0,0,0) = 0$ and $h_i(0) = 0$.

Assume that each i-th subsystem is unboundedness observable (UO) with zero offset and IOS with y_{3-i}, u_i as the inputs and y_i as the output. Specifically, there exist $\alpha_i^O \in \mathcal{K}_\infty, \beta_i \in \mathcal{KL}, \gamma_{i(3-i)} \in \mathcal{K}$, and $\gamma_i^u \in \mathcal{K}$ such that

$$|x_i(t)| \leqslant \alpha_i^O(|x_i(0)| + \|y_{3-i}\|_{[0,t]} + \|u_i\|_{[0,t]}), \tag{8}$$

$$|y_i(t)| \leqslant \max\{\beta_i(|x_i(0)|, t), \gamma_{i(3-i)}(\|y_{3-i}\|_{[0,t]}), \gamma_i^u(\|u_i\|_{[0,t]})\} \tag{9}$$

for all $t \in [0, T_{\max})$, where $[0, T_{\max})$ with $0 < T_{\max} \leqslant \infty$ is the right maximal interval for the definition of $(x_1(t), x_2(t))$.

Theorem 2 gives a small-gain result for the interconnected IOS system.

Theorem 2 Consider the interconnected system (6) and (7) satisfying (8) and (9) for $i = 1, 2$. Then the interconnected system is UO and IOS if

$$\gamma_{12} \circ \gamma_{21} < \mathrm{Id}. \tag{10}$$

Theorem 2 does not assume the forward completeness of the subsystems. Following the discussions in [10], IOS together with UO implies the forward completeness of the subsystems. If the small-gain condition is satisfied, then the forward completeness of the interconnected system is guaranteed by the IOS and UO properties of the subsystems. In [27], Theorem 2 is generalized for large-scale dynamical networks composed of more than two subsystems. This result is reviewed in Section 2.

[10] also took into account the issue of practical stability by introducing the notion of input-to-output practical stability (IOpS) property, and the small-gain theorem therein is more general than Theorem 2. Further extensions of [10] can be found in [43]. [31, 33, 44] as well as the book [22] showed the extensions of the small-gain theorem to more general complex systems such as hybrid systems and systems modeled by retarded functional differential equations.

1.2 Lyapunov-based small-gain theorem

Lyapunov functions play an irreplaceable role in the analysis and synthesis of nonlinear control systems. With the Lyapunov-based formulation of ISS, the ISS property of nonlinear systems is often tested by constructing ISS-Lyapunov functions. This subsection reviews the Lyapunov-based ISS small-gain theorem developed in [13] for interconnected ISS systems. In particular, it is shown that if an interconnected system satisfies the Lyapunov-based ISS small-gain condition, then ISS-Lyapunov functions can be constructed for the system by using the ISS-Lyapunov functions of the subsystems.

For the interconnected system (1) and (2), assume that each x_i-subsystem for $i = 1, 2$ admits a continuously differentiable ISS-Lyapunov function $V_i : \boldsymbol{R}^{n_i} \to \boldsymbol{R}_+$ satisfying the following properties:

1) There exist $\underline{\alpha}_i$, $\overline{\alpha}_i \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_i(|x_i|) \leqslant V_i(x_i) \leqslant \overline{\alpha}_i(|x_i|),\ \forall x_i; \tag{11}$$

2) There exist $\chi_{i(3-i)}$, $\chi_i^u \in \mathcal{K}$ and a continuous, positive definite α_i such that

$$V_i(x_i) \geqslant \max\{\chi_{i(3-i)}(V_{3-i}(x_{3-i})), \chi_i^u(|u_i|)\} \Rightarrow$$
$$\nabla V_i(x_i) f_i(x, u_i) \leqslant -\alpha_i(V_i(x_i)),\ \forall x,\ u_i. \tag{12}$$

Theorem 3 gives a Lyapunov formulation of the ISS small-gain theorem.

Theorem 3 Interconnected system (1) and (2) with each x_i-subsystem admitting an ISS-Lyapunov function V_i satisfying (11) and (12) is ISS if the following small-gain condition is satisfied:

$$\chi_{12} \circ \chi_{21} < \mathrm{Id}. \tag{13}$$

This section mainly focuses on continuous-time interconnected systems described by differential equations, while the counterparts of the results for discrete-time systems[45-46] and hybrid systems [31, 35, 44, 47] have also been developed based on the corresponding extensions of ISS. The interconnected hybrid systems studied in [44] may involve both stable and unstable dynamics.

If a system can be transformed into an interconnection of ISS subsystems through control design, then one may employ the ISS small-gain theorem to analyze the stability property of the closed-loop system. The gain assignment technique has been developed such that an appropriate ISS gain can be assigned to a system by means of feedback, and has been recognized to be a key step in applying the ISS small-gain theorem to nonlinear control design; e.g., [10-11, 48-49] for small-gain control designs for nonlinear uncertain systems based on the gain assignment technique.

2 Large-scale dynamical networks

The small-gain theorem introduced in Section 1 has found wide application in stability analysis, stabilization, robust adaptive control, observer design, and output regulation[50] for interconnected nonlinear systems. Although one may use the small-gain theorem recursively for interconnected systems involving more than one cycle, refined small-gain criteria are highly desirable to handle large-scale dynamical networks more efficiently.

To illustrate the need of small-gain results for network analysis, consider a nonlinear dynamical network composed of three subsystems:

$$\dot{x}_i = f_i(x),\ i = 1, 2, 3, \tag{14}$$

where $x_i \in \boldsymbol{R}^{n_i}$ is the state of the i-th subsystem, $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, x_3^{\mathrm{T}}]^{\mathrm{T}}$, and $f_i : \boldsymbol{R}^{n_1+n_2+n_3} \to \boldsymbol{R}^{n_i}$ is a locally Lipschitz function satisfying $f_i(0) = 0$.

Suppose that each x_i-subsystem has an ISS-Lyapunov function V_i, which is positive definite and radially unbounded, and satisfies

$$V_i(x_i) \geqslant \max_{j \neq i}\{\gamma_{ij}(V_j(x_j))\} \Rightarrow$$
$$\nabla V_i(x_i) f_i(x) \leqslant -\alpha_i(V_i(x_i)), \tag{15}$$

$\forall x$, where $\gamma_{ij} \in \mathcal{K} \bigcup \{0\}$ represents the ISS gains and α_i is a continuous and positive definite function. We consider the case in which only γ_{12}, γ_{13}, γ_{21}, γ_{32}, γ_{31} are nonzero ISS gains.

The gain interconnection structure of the dynamical network can be represented with a digraph, called the gain digraph, by considering the subsystems as vertices and the nonzero gain interconnections as directed links. Since the gain digraph describes the relation between the Lyapunov functions, each x_i-subsystem is represented with its ISS-Lyapunov function V_i. The gain digraph of the dynamical network defined above is shown in Figure 2.

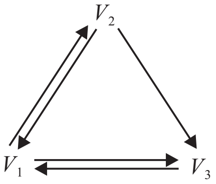

Figure 2 The gain digraph of the dynamical network (14)

We consider using the small-gain theorem introduced in Section 1 twice to analyze the stability property of the dynamical network. First, we divide the dynamical network into two parts: the (x_1, x_2)-subsystem with x_3 as the input and the x_3-subsystem with (x_1, x_2) as the input.

The (x_1, x_2)-subsystem is ISS because the small-gain condition is satisfied:

$$\gamma_{12} \circ \gamma_{21} < \mathrm{Id}. \tag{16}$$

We construct an ISS-Lyapunov function for the (x_1, x_2)-subsystem as

$$V_{(1,2)}(x_1, x_2) = \max\{V_1(x_1), \sigma(V_2(x_2))\}, \tag{17}$$

where σ is a class $\mathcal{K}_\infty$ function that is continuously differentiable on $(0, \infty)$ and satisfies

$$\sigma > \gamma_{12},\ \sigma^{-1} > \gamma_{21}. \tag{18}$$

Then, it holds that

$$\begin{aligned} &V_{(1,2)}(x_1, x_2) \geqslant \gamma_{13}(V_3(x_3)) \Rightarrow \\ &\nabla V_{(1,2)}(x_1, x_2) f_{(1,2)}(x) \leqslant \\ &-\alpha_{(1,2)}(V_{(1,2)}(x_1, x_2)),\ \text{a.e.}, \end{aligned} \tag{19}$$

where $f_{(1,2)}(x) := [f_1^{\mathrm{T}}(x), f_2^{\mathrm{T}}(x)]^{\mathrm{T}}$ and $\alpha_{(1,2)}$ is a continuous and positive definite function.

From (15), the influence of $V_{(1,2)}(x_1, x_2)$ to $V_3(x_3)$ can be represented by

$$\begin{aligned} &V_3(x_3) \geqslant \gamma_{3(1,2)}(V_{(1,2)}(x_1, x_2)) \Rightarrow \\ &\nabla V_3(x_3) f_3(x) \leqslant -\alpha_3(V_3(x_3)), \end{aligned} \tag{20}$$

where $\gamma_{3(1,2)}(s) := \max\{\gamma_{31}(s), \gamma_{32} \circ \sigma^{-1}(s)\}$ for $s \geqslant 0$.

Then, we consider the interconnection of the (x_1, x_2)-subsystem and the x_3-subsystem. The dynamical network is asymptotically stable at the origin if it satisfies the small-gain condition $\gamma_{13} \circ \gamma_{3(1,2)} < \mathrm{Id}$, or equivalently,

$$\gamma_{13} \circ \gamma_{31} < \mathrm{Id}, \tag{21}$$

$$\gamma_{13} \circ \gamma_{32} \circ \sigma^{-1} < \mathrm{Id}. \tag{22}$$

The satisfaction of condition (22) depends on the choice of σ, which is subject to constraints $\sigma > \gamma_{12}$ and $\sigma^{-1} > \gamma_{21}$. Note that (16) guarantees the existence of σ to satisfy the constraints. By choosing σ such that $\sigma^{-1} > \gamma_{21}$ and σ^{-1} is very close to γ_{21}, (22) can be guaranteed by

$$\gamma_{13} \circ \gamma_{32} \circ \gamma_{21} < \mathrm{Id}. \tag{23}$$

Thus, the dynamical network is asymptotically stable at the origin if (16), (21) and (23) are satisfied. This means that the composition of the ISS gains along every simple cycle in the gain digraph should be less than Id. This condition is referred to as the cyclic-small-gain condition.

Considering the wide interest in studying large-scale dynamical networks, it is natural to ask:

1) Is the cyclic-small-gain condition valid for general large-scale dynamical networks composed of ISS subsystems?

2) How do we construct an ISS-Lyapunov function for a dynamical network if it satisfies the cyclic-small-gain condition?

In this section, we develop cyclic-small-gain results to solve the problems for continuous-time, discrete-time, and more general hybrid dynamical networks. To make the results more accessible, we mainly consider ISS systems in this section, while some extensions to IOS systems are also provided.

2.1 Continuous-time dynamical networks

Consider the following large-scale dynamical network containing N subsystems:

$$\dot{x}_i = f_i(x, u_i),\ i = 1, 2, \cdots, N, \tag{24}$$

where $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \cdots, x_N^{\mathrm{T}}]^{\mathrm{T}}$ with $x_i \in \boldsymbol{R}^{n_i}$ is the state, $u_i \in \boldsymbol{R}^{m_i}$ represents the external inputs, and each $f_i : \boldsymbol{R}^{n+m_i} \to \boldsymbol{R}^{n_i}$ with $n = \sum\limits_{j=1}^{N} n_j$ is a locally Lipschitz function satisfying $f_i(0, 0) = 0$. The external input $u = [u_1^{\mathrm{T}}, u_2^{\mathrm{T}}, \cdots, u_N^{\mathrm{T}}]^{\mathrm{T}}$ is a measurable and locally essentially bounded function from $\boldsymbol{R}_+$ to $\boldsymbol{R}^m$ with $m = \sum\limits_{i=1}^{N} m_i$. Denote $f(x, u) = [f_1^{\mathrm{T}}(x, u_1), f_2^{\mathrm{T}}(x, u_2), \cdots, f_N^{\mathrm{T}}(x, u_N)]^{\mathrm{T}}$.

Assume that for $i = 1, 2, \cdots, N$, each x_i-subsystem admits a continuously differentiable ISS-Lyapunov function $V_i : \boldsymbol{R}^{n_i} \to \boldsymbol{R}_+$ satisfying the following properties:

1) There exist $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_i(|x_i|) \leqslant V_i(x_i) \leqslant \overline{\alpha}_i(|x_i|),\ \forall x_i; \tag{25}$$

2) There exist $\gamma_{ij} \in \mathcal{K} \bigcup \{0\}$ $(j = 1, 2, \cdots, N,\ j \neq i)$ and $\gamma_{ui} \in \mathcal{K} \bigcup \{0\}$ such that

$$V_i(x_i) \geqslant \max_{j \neq i}\{\gamma_{ij}(V_j(x_j)), \gamma_{ui}(|u_i|)\} \Rightarrow$$

$$\nabla V_i(x_i) f_i(x, u_i) \leqslant -\alpha_i(V_i(x_i)),\ \forall x,\ u_i, \tag{26}$$

where α_i is a continuous and positive definite function.

For systems that are formulated in the dissipation form, property 2) above should be replaced by

2′) There exist $\alpha'_i \in \mathcal{K}_\infty, \sigma'_{ij} \in \mathcal{K} \bigcup \{0\}$ $(j = 1, 2, \cdots, N,\ j \neq i)$ and $\sigma'_{ui} \in \mathcal{K} \bigcup \{0\}$ such that

$$\nabla V_i(x_i) f_i(x, u_i) \leqslant$$

$$-\alpha'_i(V_i(x_i)) + \max\{\sigma'_{ij}(V_j(x_j)), \sigma'_{ui}(|u_i|)\}. \tag{27}$$

Due to the equivalence of the two forms for continuous-time systems, we only consider the gain margin form in the following discussions.

By considering the subsystems as vertices and the nonzero gain interconnections as directed links, the gain interconnection structure of the dynamical network can be represented by a digraph, called the gain digraph. Then, concepts from graph theory, such as path, reachability, and simple cycle, can be used to describe the gain interconnections in the dynamical network. Since the gains are defined with Lyapunov functions, for $i = 1, 2, \cdots, N$, each x_i-subsystem is represented by its Lyapunov function V_i.

Theorem 4 answers the question on the validity of the cyclic-small-gain condition for continuous-time large-scale dynamical networks with subsystems admitting ISS-Lyapunov functions.

Theorem 4 Consider the continuous-time dynamical network (24) with each x_i-subsystem admitting a continuously differentiable ISS-Lyapunov function V_i satisfying (25) and (26). Then, it is ISS with x as the state and u as the input if for every simple cycle $(V_{i_1}, V_{i_2}, \cdots, V_{i_r}, V_{i_1})$ in the gain digraph,

$$\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \cdots \circ \gamma_{i_r i_1} < \mathrm{Id}, \tag{28}$$

where $r = 2, 3, \cdots, N$ and $1 \leqslant i_j \leqslant N, i_j \neq i_{j'}$ if $j \neq j'$.

2.1.1 Basic idea of constructing ISS-Lyapunov functions

The small-gain theorem introduced in Section 1 considers the case in which dynamical network (24) contains two subsystems, i.e., $N = 2$. In this case, if $\gamma_{12} \circ \gamma_{21} < \mathrm{Id}$, then the dynamical network is ISS and an ISS-Lyapunov function can be constructed as

$$V(x) = \max\{V_1(x_1), \sigma(V_2(x_2))\}, \tag{29}$$

where $\sigma \in \mathcal{K}_\infty$ is continuously differentiable on $(0, \infty)$ and satisfies

$$\sigma > \gamma_{12},\ \sigma^{-1} > \gamma_{21}. \tag{30}$$

Recall the fact that $\gamma_{12} \circ \gamma_{21} < \mathrm{Id} \Leftrightarrow \gamma_{21} \circ \gamma_{12} < \mathrm{Id}$. By using Lemma C.1[51] twice, there exist $\hat{\gamma}_{12}, \hat{\gamma}_{21} \in \mathcal{K}_\infty$ which are continuously differentiable on $(0, \infty)$ and satisfy $\hat{\gamma}_{12} > \gamma_{12}, \hat{\gamma}_{21} > \gamma_{21}$ and $\hat{\gamma}_{12} \circ \hat{\gamma}_{21} < \mathrm{Id}$. Thus, with γ_{12}, γ_{21} replaced by $\hat{\gamma}_{12}, \hat{\gamma}_{21}$ (as shown in Figure 3), the small-gain condition is still satisfied.

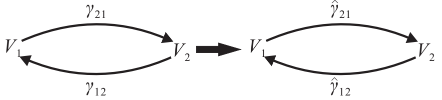

Figure 3 The replacement of the ISS gains

If we choose $\sigma = \hat{\gamma}_{12}$, then condition (30) is satisfied and the resulting ISS-Lyapunov function is

$$V(x) = \max\{V_1(x_1), \hat{\gamma}_{12}(V_2(x_2))\}. \tag{31}$$

Since $\hat{\gamma}_{12}$ is a modification of the ISS gain γ_{12}, the term $\hat{\gamma}_{12}(V_2(x_2))$ can be considered as the “potential influence” of V_2 acting on V_1 with modified gain $\hat{\gamma}_{12}$.

2.1.2 A class of ISS-Lyapunov functions for dynamical networks

Based on the idea of potential influence, a class of ISS-Lyapunov functions are constructed for large-scale dynamical networks satisfying the cyclic-small-gain condition.

Recall the fact that for any $\chi_1, \chi_2 \in \mathcal{K} \bigcup \{0\}, \chi_1 \circ \chi_2 < \mathrm{Id} \Leftrightarrow \chi_2 \circ \chi_1 < \mathrm{Id}$. Consider a dynamical network in the form of (24) with the cyclic-small-gain condition (28) satisfied. For each $i^* = 1, 2, \cdots, N$, it holds that

$$\gamma_{i^* i_2} \circ \gamma_{i_2 i_3} \circ \cdots \circ \gamma_{i_r i^*} < \mathrm{Id} \tag{32}$$

for $r = 2, 3, \cdots, N, 1 \leqslant i_j \leqslant N, i_j \neq i^*, i_j \neq i_{j'}$ if $j \neq j'$. By using Lemma C.1[51], if $\gamma_{i^* i_2} \neq 0$, then one can find a $\hat{\gamma}_{i^* i_2} \in \mathcal{K}_\infty$ which is continuously differentiable on $(0, \infty)$ and satisfies $\hat{\gamma}_{i^* i_2} > \gamma_{i^* i_2}$ such that (32) still holds with $\gamma_{i^* i_2}$ replaced by $\hat{\gamma}_{i^* i_2}$.

By repeating this procedure for all the $\gamma_{i^* i_2}$ with $i^* = 1, 2, \cdots, N, i_2 \neq i^*$, there exist $\hat{\gamma}_{(\cdot)}$'s such that

1) $\hat{\gamma}_{(\cdot)} \in \mathcal{K}_\infty$ and $\hat{\gamma}_{(\cdot)} > \gamma_{(\cdot)}$ if $\gamma_{(\cdot)} \in \mathcal{K}$; $\hat{\gamma}_{(\cdot)} = 0$ if $\gamma_{(\cdot)} = 0$.

2) $\hat{\gamma}_{(\cdot)}$'s are continuously differentiable on $(0, \infty)$.

3) For each $r = 2, 3, \cdots, N$,

$$\hat{\gamma}_{i_1 i_2} \circ \cdots \circ \hat{\gamma}_{i_r i_1} < \mathrm{Id} \tag{33}$$

holds for all $1 \leqslant i_j \leqslant N$ and $i_j \neq i_{j'}$ if $j \neq j'$.

Through the approach above, all the nonzero gains in the dynamical network are replaced by the $\hat{\gamma}_{(\cdot)}$'s, which are of class $\mathcal{K}_\infty$ and continuously differentiable on $(0, \infty)$ such that the cyclic-small-gain condition is still satisfied. Note that the replacement of the nonzero gains does not influence the gain digraph.

In the large-scale dynamical network, the potential influence acting on the p-th subsystem from all the subsystems can be described as

$$\boldsymbol{V}^{[p]} = \bigcup_{j=1,2,\cdots,N} \boldsymbol{V}_j^{[p]}(x) \tag{34}$$

with

$$\boldsymbol{V}_j^{[p]}(x) = \{\hat{\gamma}_{i_1^{[p]} i_2^{[p]}} \circ \cdots \circ \hat{\gamma}_{i_{j-1}^{[p]} i_j^{[p]}}(V_{i_j^{[p]}}(x_{i_j^{[p]}}))\},$$

where $i_1^{[p]} = p, i_k^{[p]} \in \{1,2,\cdots,N\}, k \in \{1,2,\cdots,j\}, i_k^{[p]} \neq i_{k'}^{[p]}$ if $k \neq k'$, for $j = 1,2,\cdots,N$. Clearly, each element in $\boldsymbol{V}_j^{[p]}(x)$ corresponds to a simple path ending at V_p in the gain digraph.

Note that $\hat{\gamma}_{(\cdot)} \in \mathcal{K}_\infty \bigcup \{0\}$. It is easy to verify that $\max \boldsymbol{V}^{[p]}$ is positive definite and radially unbounded with respect to the Lyapunov functions of the subsystems with indices belonging to $\mathcal{RS}(p)$.

Correspondingly, the potential influence of the external input $u = [u_1^{\mathrm{T}}, u_2^{\mathrm{T}}, \cdots, u_N^{\mathrm{T}}]^{\mathrm{T}}$ acting on the p-th subsystem can be described as

$$\boldsymbol{U}^{[p]} = \bigcup_{j=1,2,\cdots,N} \boldsymbol{U}_j^{[p]} \tag{35}$$

with

$$\boldsymbol{U}_j^{[p]} = \{\hat{\gamma}_{i_1^{[p]} i_2^{[p]}} \circ \cdots \circ \hat{\gamma}_{i_{j-1}^{[p]} i_j^{[p]}} \circ \gamma_{u i_j^{[p]}}(|u_{i_{j-1}^{[p]}}|)\} \tag{36}$$

for $j = 1,2,\cdots,N$.

Define

$$V_\Pi(x) = \max \boldsymbol{V}_\Pi(x) = \max(\bigcup_{p\in\Pi} \boldsymbol{V}^{[p]}(x)), \tag{37}$$

where set $\Pi \subseteq \{1,2,\cdots,N\}$ satisfies $\bigcup_{p\in\Pi}(\mathcal{RS}(p)) = \{1,2,\cdots,N\}$.

It can be directly verified that $\max(\bigcup_{p\in\Pi} \boldsymbol{V}^{[p]})$ is positive definite and radially unbounded with respect to $\max\{V_1, V_2, \cdots, V_N\}$ and thus with respect to x, i.e., there exist $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}(|x|) \leqslant V_\Pi(x) \leqslant \overline{\alpha}(|x|)$ for all x. It can also be observed that V_Π is locally Lipschitz on $\boldsymbol{R}^n \backslash \{0\}$. Thanks to Rademacher's theorem (e.g., [52] p.216), V_Π is differentiable almost everywhere.

Correspondingly, denote

$$u_\Pi = \max \boldsymbol{U}_\Pi = \max(\bigcup_{p\in\Pi} \boldsymbol{U}^{[p]}). \tag{38}$$

It can be verified that there exists a $\gamma^u \in \mathcal{K}_\infty$ such that $u_\Pi \leqslant \gamma^u(|u|)$ for all u.

Subsection 3.13 in [51] shows that $V_\Pi(x)$ is an ISS-Lyapunov function (not necessarily continuously differentiable) of the dynamical network with u_Π as the new input.

2.2 Discontinuous dynamical networks

In [38], the concepts of ISS and the ISS-Lyapunov function are extended to discontinuous systems and also an extended Filippov solution is proposed for interconnected discontinuous systems by using differential inclusions. Based on the concept of extended Filippov solution, an ISS small-gain theorem has been developed for discontinuous systems. Based on the results in [38], we develop a cyclic-small-gain theorem for discontinuous dynamical networks with the subsystems represented by differential inclusions:

$$\dot{x}_i \in F_i(x, u_i),\ i = 1,2,\cdots,N, \tag{39}$$

where $F_i : \boldsymbol{R}^{n+m_i} \rightsquigarrow \boldsymbol{R}^{n_i}$ is a convex, compact, and upper semi-continuous set-valued map satisfying $0 \in F_i(0,0)$, and the variables are defined in the same way as for (24).

Assume that each x_i-subsystem in (39) admits an ISS-Lyapunov function V_i satisfying (25) and

$$V_i(x_i) \geqslant \max_{j=1,2,\cdots,N;\, j\neq i}\{\gamma_{ij}(V_j(x_j)), \gamma_i^u(|u_i|)\} \Rightarrow$$
$$\max_{f_i\in F_i(x,u_i)} \nabla V_i(x_i) f_i \leqslant -\alpha_i(V_i(x_i)) \tag{40}$$

wherever ∇V_i exists. Clearly, (40) is a direct modification of (26).

We have such a cyclic-small-gain result for discontinuous dynamical networks: if the cyclic-small-gain condition (28) is satisfied, then the discontinuous dynamical network is ISS and an ISS-Lyapunov function V_Π can be constructed as in (37) such that

$$V_\Pi(x) \geqslant u_\Pi \Rightarrow \max_{f\in F(x,u)} \nabla V_\Pi(x) f \leqslant -\alpha_\Pi(V_\Pi(x)) \tag{41}$$

wherever ∇V exists, with $F(x,u) = [F_1^{\mathrm{T}}(x,u_1), F_2^{\mathrm{T}}(x,u_2), \cdots, F_N^{\mathrm{T}}(x,u_N)]^{\mathrm{T}}$.

2.3 Dynamical networks of IOS subsystems

Corresponding to Theorem 2, this subsection presents cyclic-small-gain results for large-scale dynamical networks composed of IOS subsystems. Time-delay issues are also discussed. Consider a large-scale dynamical network in the form of

$$\dot{x}_1 = f_1(x_1, y_2, y_3, \cdots, y_n, u_1), \tag{42}$$

$$\dot{x}_2 = f_2(x_2, y_1, y_3, \cdots, y_n, u_2), \tag{43}$$

$$\vdots$$

$$\dot{x}_n = f_n(x_n, y_1, y_2, \cdots, y_{n-1}, u_n) \tag{44}$$

with output maps

$$y_i = h_i(x_i),\ i = 1,2,\cdots,n. \tag{45}$$

For each i-th subsystem, $x_i \in \boldsymbol{R}^{n_i}$ is the state, $u_i \in \boldsymbol{R}^{m_i}$ is the input, $y_i \in \boldsymbol{R}^{l_i}$ is the output, and f_i, h_i are locally Lipschitz functions. Denote $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \cdots, x_n^{\mathrm{T}}]^{\mathrm{T}}, y = [y_1^{\mathrm{T}}, y_2^{\mathrm{T}}, \cdots, y_n^{\mathrm{T}}]^{\mathrm{T}}$, and $u =$

$[u_1^{\mathrm{T}}, u_2^{\mathrm{T}}, \cdots, u_n^{\mathrm{T}}]^{\mathrm{T}}$. By considering u as a function of time, assume that u is measurable and locally essentially bounded.

Suppose that each i-th subsystem is UO with zero offset and IOS with y_j for $j \neq i$ and u_i as the inputs and y_i as the output. Specifically, there exist $\alpha_i^O \in \mathcal{K}_\infty, \beta_i \in \mathcal{KL}, \gamma_{ij} \in \mathcal{K}$ and $\gamma_i^u \in \mathcal{K}$ such that

$$|x_i(t)| \leqslant \alpha_i^O\Big(|x_i(0)| + \sum_{j\neq i} \|y_j\|_{[0,t]} + \|u_i\|_{[0,t]}\Big), \tag{46}$$

$$|y_i(t)| \leqslant \max_{j\neq i}\{\beta_i(|x_i(0)|, t), \gamma_{ij}(\|y_j\|_{[0,t]}), \gamma_i^u(\|u_i\|_\infty)\} \tag{47}$$

for all $t \in [0, T_{\max})$, where $[0, T_{\max})$ with $0 < T_{\max} \leqslant \infty$ is the right maximal interval for the definition of $(x_1(t), x_2(t), \cdots, x_n(t))$.

A cyclic-small-gain theorem for large-scale dynamical networks composed of IOS subsystems is given in Theorem 5.

Theorem 5 Consider dynamical network (42) ∼ (45) satisfying (46) and (47) for $i = 1, 2, \cdots, n$. Then the dynamical network is UO and IOS if the cyclic-small-gain condition (28) is satisfied.

In [27], a cyclic-small-gain theorem is presented for large-scale dynamical networks composed of output-Lagrange input-to-output stable (OLIOS) subsystems with an induction-based proof. For the OLIOS systems, the cyclic-small-gain theorem can be proved by using the equivalence between OLIOS and the conjunction of uniform bounded-input bounded-state stability (UBIBS) and the output asymptotic gain property. But this method seems not directly applicable to the systems with only UO and IOS properties. For Theorem 5, Appendix D.3[51] presents a sketch of a proof, which can be considered as a combination of the methods in [10] and [27].

The cyclic-small-gain condition is also valid for the large-scale dynamical networks with interconnection time delays. This topic has been studied in [22, 42]. Consider a dynamical network in the following form:

$$\begin{aligned}\dot{x}_1(t) = {} & f_1(x_1(t), y_2(t-\tau_{12}), y_3(t-\tau_{13}), \\ & \cdots, y_n(t-\tau_{1n}), u_1(t)),\end{aligned} \tag{48}$$

$$\begin{aligned}\dot{x}_2(t) = {} & f_2(x_2(t), y_1(t-\tau_{21}), y_3(t-\tau_{23}), \\ & \cdots, y_n(t-\tau_{2n}), u_2(t)),\end{aligned} \tag{49}$$

$$\vdots$$

$$\begin{aligned}\dot{x}_n(t) = {} & f_n(x_n(t), y_1(t-\tau_{n1}), y_2(t-\tau_{n2}), \\ & \cdots, y_{n-1}(t-\tau_{n(n-1)}), u_n(t))\end{aligned} \tag{50}$$

with output maps defined in (45), where $\tau_{ij} : \boldsymbol{R}_+ \to [0, \theta]$ for $i \neq j$ represents the time delay of the interconnection from the j-th subsystem to the i-th subsystem with constant $\theta \geqslant 0$ being the largest time delay. The analogous definitions of UO and IOS for systems with delays can be found in [42].

Intuitively, since

$$|y_i(t - \tau_{ji})| \leqslant \|y_i\|_{[-\theta,\infty)}, \tag{51}$$

one may consider the time-delay components (shown in Figure 4) as subsystems with the identity gain, so they should not cause violation of the cyclic-small-gain condition for a system when introduced. This way of thinking explains why small-gain based controller designs often automatically guarantee robustness to arbitrarily large (communication-induced state) delays.

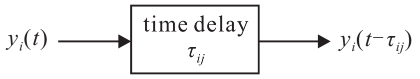

Figure 4 A time-delay component

Theorem 6 gives a cyclic-small-gain result for large-scale dynamical networks with time-delays.

Theorem 6 Consider dynamical network (48) ∼ (50) with output maps defined by (45). Suppose that if the time delays do not exist, i.e., $\theta = 0$, each i-th subsystem with $i = 1, 2, \cdots, n$ satisfies (46) and (47). Then the dynamical network with $\theta \geqslant 0$ is UO and IOS if the cyclic-small-gain condition (28) is satisfied.

2.4 Discrete-time dynamical networks

In view of the critical importance of discrete-time system theory in computer-aided control engineering applications, in this subsection, we generalize the ISS cyclic-small-gain theorem introduced in Subsection 2.1 to discrete-time dynamical networks. Due to phenomena which are particular to discrete-time systems (see Example 1 below), such a generalization is nontrivial.

Analogous to the continuous-time dynamical network studied in Subsection 2.1, the discrete-time dynamical network addressed in this subsection is composed of N discrete-time subsystems in the following form:

$$x_i(T+1) = f_i(x(T), u_i(T)),\ i = 1, 2, \cdots, N, \tag{52}$$

where $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \cdots, x_N^{\mathrm{T}}]^{\mathrm{T}}$ with $x_i \in \boldsymbol{R}^{n_i}$ is the state, $u_i \in \boldsymbol{R}^{n_{ui}}$ represent the external inputs, and $f_i : \boldsymbol{R}^{n+n_{ui}} \to \boldsymbol{R}^{n_i}$ with $n := \sum_{i=1}^{N} n_i$ is continuous. T takes values in $\boldsymbol{Z}_+$. It is assumed that $f_i(0, 0) = 0$, and the external input $u = [u_1^{\mathrm{T}}, u_2^{\mathrm{T}}, \cdots, u_N^{\mathrm{T}}]^{\mathrm{T}}$ is bounded. Denote $f(x, u) = [f_1^{\mathrm{T}}(x, u_1), f_2^{\mathrm{T}}(x, u_2), \cdots, f_N^{\mathrm{T}}(x, u_N)]^{\mathrm{T}}$.

There are two kinds of Lyapunov formulations for discrete-time ISS systems: the dissipation form and the gain margin form. We first give the dissipation form.

For $i = 1, 2, \cdots, N$, each x_i-subsystem admits a continuous ISS-Lyapunov function $V_i : \boldsymbol{R}^{n_i} \to \boldsymbol{R}_+$ satisfying the following:

1) There exist $\underline{\alpha}_i$, $\overline{\alpha}_i \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_i(|x_i|) \leqslant V_i(x_i) \leqslant \overline{\alpha}_i(|x_i|),\ \forall x_i; \tag{53}$$

2) There exist $\alpha_i \in \mathcal{K}_\infty, \sigma_{ij} \in \mathcal{K}\bigcup\{0\}$ and $\sigma_{ui} \in \mathcal{K}\bigcup\{0\}$ such that

$$\begin{aligned}&V_i(f_i(x, u_i)) - V_i(x_i) \leqslant \\ &-\alpha_i(V_i(x_i)) + \max_{j \neq i}\{\sigma_{ij}(V_j(x_j)), \sigma_{ui}(|u_i|)\},\ \forall x,\ u_i.\end{aligned} \tag{54}$$

Without loss of generality, we assume $(\mathrm{Id} - \alpha_i) \in \mathcal{K}$. Note that if $(\mathrm{Id} - \alpha_i) \notin \mathcal{K}$, one can always find an $\alpha_i' < \alpha_i$ such that $(\mathrm{Id} - \alpha_i') \in \mathcal{K}$ and property (54) holds with α_i replaced by α_i'. Then, we consider

$$\hat{\gamma}_{ij} = \alpha_i^{-1} \circ (\mathrm{Id} - \rho_i)^{-1} \circ \sigma_{ij} \tag{55}$$

as the ISS gain from V_j to V_i, with ρ_i being a continuous and positive definite function and satisfying $(\mathrm{Id} - \rho_i) \in \mathcal{K}_\infty$.

Correspondingly, the ISS gain from the external input u_i to V_i is defined as

$$\hat{\gamma}_{ui} = \alpha_i^{-1} \circ (\mathrm{Id} - \rho_i)^{-1} \circ \sigma_{ui}. \tag{56}$$

The gain margin formulation of the ISS-Lyapunov functions can be obtained by modifying property 2) as follows: there exist functions α_i' (continuous and positive definite) and $\gamma_{ij}', \gamma_{ui}' \in \mathcal{K}\bigcup\{0\}$ such that

$$\begin{aligned}&V_i(x_i) \geqslant \max_{j \neq i}\{\gamma_{ij}'(V_j(x_j)), \gamma_{ui}'(|u_i|)\} \Rightarrow \\ &V_i(f_i(x, u_i)) - V_i(x_i) \leqslant -\alpha_i'(V_i(x_i)),\ \forall x,\ u_i.\end{aligned} \tag{57}$$

In contrast to the continuous-time systems discussed in Subsection 2.1, the trajectories of a discrete-time system may “jump out” of the region determined by the gain margin in (57), which means that the γ_{ij}' and the γ_{ui}' in (57) may not be the true ISS gains. Consider Example 1.

Example 1　Consider a discrete-time system

$$z(T+1) = g(z(T), |w(T)|), \tag{58}$$

where $z \in \boldsymbol{R}$ is the state, $w \in \boldsymbol{R}^m$ is the external input, and $g : \boldsymbol{R}^{m+1} \to \boldsymbol{R}$ defined in Figure 5 is continuous. Define $V_z(z) = |z|$. Then, one can find a small $\delta > 0$ such that

$$\begin{aligned}&V_z(z) \geqslant (1+\delta)|w| \Rightarrow \\ &V_z(g(z, |w|)) - V_z(z) \leqslant -\alpha_z(V_z(z)),\end{aligned} \tag{59}$$

where α_z is a continuous and positive definite function. Then, $(1+\delta)$ is the “ISS gain” defined by the gain margin formulation (57).

However, from Figure 5, it is possible that $V_z(g(z, |w|)) > (1+\delta)|w|$, even if $V_z(z) \leqslant (1+\delta)|w|$, which means that the state of the discrete-time nonlinear system may “jump out” of the region defined by the gain margin. To solve this problem, one may find an $\alpha_g \in \mathcal{K}$ such that $|g(z, |w|)| \leqslant \alpha_g(|w|)$ whenever $|z| \leqslant \alpha_g(|w|)$, and define $\gamma_w(s) = \max\{(1+\delta)s, \alpha_g(s)\}$ for $s \geqslant 0$. Then, the phenomenon of “jump out” can be avoided as

$$\begin{aligned}&V_z(z) \geqslant \gamma_w(|w|) \Rightarrow \\ &V_z(g(z, |w|)) - V_z(z) \leqslant -\alpha_z(V_z(z)),\end{aligned} \tag{60}$$

$$\begin{aligned}&V_z(z) \leqslant \gamma_w(|w|) \Rightarrow \\ &V_z(g(z, |w|)) \leqslant \gamma_w(|w|),\end{aligned} \tag{61}$$

and γ_w can be used as the ISS gain for the discrete-time system.

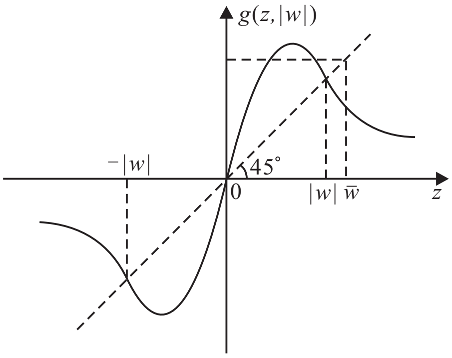

Figure 5　**An example of the gain margin property of discrete-time systems, where $\bar{w} = (1+\delta)|w|$**

Following the idea in Example 1, to take into account the “jump out” issue, the gain margin formulation (57) is consolidated with

$$\begin{aligned}&V_i(x_i) \leqslant \max_{j \neq i}\{\gamma_{ij}'(V_j(x_j)), \gamma_{ui}'(|u_i|)\} \Rightarrow \\ &V_i(f_i(x, u_i)) \leqslant \\ &(\mathrm{Id} - \delta_i') \times (\max_{j \neq i}\{\gamma_{ij}'(V_j(x_j)), \gamma_{ui}'(|u_i|)\}),\end{aligned} \tag{62}$$

where δ_i' is a continuous and positive definite function satisfying $(\mathrm{Id} - \delta_i') \in \mathcal{K}_\infty$.

By combining (57) and (62), the refined gain margin formulation is described with property 1) above and

2′) There exist $\hat{\gamma}_{ij} \in \mathcal{K}\bigcup\{0\}$ and $\hat{\gamma}_{ui} \in \mathcal{K}\bigcup\{0\}$ such that

$$\begin{aligned}&V_i(f_i(x, u_i)) \leqslant \\ &(\mathrm{Id} - \delta_i)(\max_{j \neq i}\{\hat{\gamma}_{ij}(V_j(x_j)), V_i(x_i), \hat{\gamma}_{ui}(|u_i|)\})\end{aligned} \tag{63}$$

for all x, u_i, where δ_i is a continuous and positive definite function satisfying $(\mathrm{Id} - \delta_i) \in \mathcal{K}_\infty$.

We present the cyclic-small-gain results for discrete-time dynamical networks formulated in the dissipation form and the gain margin form in Theorems 7 and 8, respectively.

Theorem 7 Consider the discrete-time dynamical network (52) with each x_i-subsystem having a continuous ISS-Lyapunov function V_i satisfying (53) and (54). Then, it is ISS with x as the state and u as the input, if there exist continuous and positive definite functions ρ_i satisfying $(\mathrm{Id}-\rho_i)\in\mathcal{K}_\infty$ such that for every simple cycle $(V_{i_1}, V_{i_2}, \cdots, V_{i_r}, V_{i_1})$ in the gain digraph,

$$\hat{\gamma}_{i_1i_2}\circ\hat{\gamma}_{i_2i_3}\circ\cdots\circ\hat{\gamma}_{i_ri_1}<\mathrm{Id}, \tag{64}$$

where $r=2,3,\cdots,N$ and $1\leqslant i_j\leqslant N, i_j\neq i_{j'}$ if $j\neq j'$.

Theorem 8 Consider the discrete-time dynamical network (52) with each x_i-subsystem having a continuous ISS-Lyapunov function V_i satisfying (53) and (63). Then, the dynamical network is ISS with x as the state and u as the input if for every simple cycle $(V_{i_1}, V_{i_2}, \cdots, V_{i_r}, V_{i_1})$ in the gain digraph,

$$\hat{\gamma}_{i_1i_2}\circ\hat{\gamma}_{i_2i_3}\circ\cdots\circ\hat{\gamma}_{i_ri_1}<\mathrm{Id}, \tag{65}$$

where $r=2,3,\cdots,N, 1\leqslant i_j\leqslant N, i_j\neq i_{j'}$ if $j\neq j'$.

As for continuous-time dynamical networks, Theorems 7 and 8 are proved by constructing ISS-Lyapunov functions. The ISS-Lyapunov function candidates for discrete-time dynamical networks are constructed like continuous-time dynamical networks in Subsection 2.1.2. One difference is that the ISS -Lyapunov functions constructed for discrete-time systems are only required to be continuous.

2.5 Hybrid dynamical networks

Based on the results for continuous-time dynamical networks and discrete-time dynamical networks, it is possible to develop a cyclic-small-gain result for hybrid dynamical networks, which involve both continuous-time and discrete-time dynamics. The hybrid dynamical network studied in this subsection is composed of N subsystems whose trajectories may be continuous, piecewise constant, or impulsive on the timeline. Define $\mathcal{N}=\{1,2,\cdots,N\}$ as the set of indices of the subsystems. For $i\in\mathcal{N}$, each i-th subsystem of the dynamical network is modeled by

$$\dot{x}_i(t)=f_i(x(t),u_i(t)),\ t\in\boldsymbol{R}_+\backslash\pi_i; \tag{66}$$

$$x_i(t)=g_i(x(t^-),u_i(t^-)),\ t\in\pi_i, \tag{67}$$

where $x_i\in\boldsymbol{R}^{n_i}$ is the state of the i-th subsystem, $x=[x_1^{\mathrm{T}},x_2^{\mathrm{T}},\cdots,x_N^{\mathrm{T}}]^{\mathrm{T}}\in\boldsymbol{R}^n$ with $n:=\sum\limits_{i=1}^N n_i$ is the state of the dynamical network, $u_i:\boldsymbol{R}_+\to\boldsymbol{R}^{m_i}$ is the input of the i-th subsystem, and $\pi_i\subset\boldsymbol{R}_+$ is the set of impulsive time instants of the i-th subsystem. For each $i\in\mathcal{N}$, it is assumed that $f_i:\boldsymbol{R}^{n+m_i}\to\boldsymbol{R}^{n_i}$ is locally Lipschitz and $f_i(0,0)=0; g_i:\boldsymbol{R}^{n+m_i}\to\boldsymbol{R}^{n_i}$ is continuous and $g_i(0,0)=0$. Denote $u=[u_1^{\mathrm{T}},u_2^{\mathrm{T}},\cdots,u_N^{\mathrm{T}}]^{\mathrm{T}}$ as the input vector of the hybrid dynamical network. Assume that each u_i is piecewise continuous and bounded.

Note that differential equation (66) represents continuous-time dynamics, and difference equation (67) captures discrete-time dynamics. We consider three kinds of subsystems: the first kind is described by only continuous-time models (66) with $\pi_i=\varnothing$, the second kind is described by only discrete-time models (67), and the third kind is described by a mix of (66) and (67) with $\pi_i\neq\varnothing$. Assumption 1 is made on the ISS properties of the subsystems.

Assumption 1 Each x_i-subsystem for $i\in\mathcal{N}$ has Lyapunov-based ISS properties. Specifically, for $i\in\mathcal{N}$, there exists a function $V_i:\boldsymbol{R}^{n_i}\to\boldsymbol{R}_+$ which is locally Lipschitz on $\boldsymbol{R}^{n_i}\backslash\{0\}$, positive definite and radially unbounded, and satisfies the following:

1) For each $i\in\mathcal{N}_C, \pi_i=\varnothing$, and there exist $\gamma_{ij},\gamma_{u_i}\in\mathcal{K}\bigcup\{0\}$ with $j\neq i$ such that

$$V_i(x_i)\geqslant\max_{j\neq i}\{\gamma_{ij}(V_j(x_j)),\gamma_{u_i}(|u_i|)\}\Rightarrow$$
$$\nabla V_i(x_i)f_i(x,u_i)\leqslant-\alpha_i(V_i(x_i)),\ \text{a.e.}, \tag{68}$$

where α_i is a continuous and positive definite function;

2) For each $i\in\mathcal{N}_D, f_i\equiv0$, and there exist $\gamma_{ij},\gamma_{u_i}\in\mathcal{K}\bigcup\{0\}$ with $j\neq i$ such that

$$V_i(g_i(x,u_i))\leqslant$$
$$(\mathrm{Id}-\rho_i)(\max_{j\neq i}\{\gamma_{ij}(V_j(x_j)),V_i(x_i),\gamma_{u_i}(|u_i|)\}), \tag{69}$$

where ρ_i is a continuous and positive definite function and satisfies $(\mathrm{Id}-\rho_i)\in\mathcal{K}_\infty$;

3) For each $i\in\mathcal{N}_H, \pi_i\neq\varnothing$, and there exist $\gamma_{ij},\gamma_{u_i}\in\mathcal{K}\bigcup\{0\}$ with $j\neq i$ such that both properties (68) and (69) are satisfied.

It should be noted that, if the π_i's are different, then the impulsive time instants of different subsystems are different. From this point of view, the hybrid dynamical network (66) and (67) is more general than the discrete-time dynamical network (52), even if $\mathcal{N}_C\bigcup\mathcal{N}_H=\varnothing$.

A mild assumption is made on the intervals between the impulsive time instants. For each $i\in\mathcal{N}_D, \pi_i$ is of the form $\pi_i=\{t_{iw}>0: w\in\boldsymbol{Z}_+\}$, and there exist constants $\overline{\delta t},\underline{\delta t}>0$ such that for all $i\in\mathcal{N}_D$,

$$\underline{\delta t}\leqslant t_{i(w+1)}-t_{iw}\leqslant\overline{\delta t} \tag{70}$$

holds for all $w \in \boldsymbol{Z}_+$.

If for each i-th subsystem with $i \in \mathcal{N}_D$, there are an upper bound and a lower bound of the intervals between the impulsive time instants, then one can always find common bounds for all the discrete-time subsystems.

Under the conditions above, the existence and uniqueness of solutions of the dynamical network with subsystems in the form of (66) and (67) can be guaranteed in the sense of Carathéology[53]. We use $x(t,t_0,\xi,u) = [x_1^{\mathrm{T}}(t,t_0,\xi,u), x_2^{\mathrm{T}}(t,t_0,\xi,u),\cdots,x_N^{\mathrm{T}}(t,t_0,\xi,u)]^{\mathrm{T}}$ or simply $x(t)$ to denote the state trajectory of the dynamical network with initial condition $\xi \in \boldsymbol{R}^n$ at time t_0 and input u. For each $i \in \mathcal{N}_D \bigcap \mathcal{N}_H, x_i(t,t_0,\xi,u)$ is right-continuous on the timeline. It should be noted that the semi-group property is satisfied for the hybrid dynamical network because the impulsive time sets π_i's are fixed and do not depend on the initial condition[33].

The main result of this section is that a hybrid dynamical network composed of subsystems (66) and (67) is ISS if it satisfies the cyclic-small-gain condition.

2.5.1 Equivalence between cyclic-small-gain and gains less than the identity

The proofs of the cyclic-small-gain theorems for continuous-time and discrete-time dynamical networks mainly deal with the simple cycles in the gain digraphs. For hybrid dynamical networks, the analysis of the cycles involving both continuous-time and discrete-time dynamics could be much more complicated. In this subsection, a result on the equivalence between cyclic-small-gain and gains less than Id is developed. Based on this observation, in the following subsection, the cyclic-small-gain theorem for hybrid dynamical networks is proved by showing that hybrid dynamical networks with interconnection gains less than Id are ISS.

The proof of the equivalence is based on the fact that, for a continuous-time or discrete-time system, if V is an ISS-Lyapunov function, then for any $\sigma \in \mathcal{K}_\infty$ being locally Lipschitz on $(0,\infty), \sigma(V)$ is also an ISS-Lyapunov function. Consider a continuous-time system, for example. Assume that system $\dot{x} = f(x,u)$ with state $x \in \boldsymbol{R}^n$ and external input $u \in \boldsymbol{R}^m$ is ISS with $V: \boldsymbol{R}^n \to \boldsymbol{R}_+$ as an ISS-Lyapunov function satisfying

$$V(x) \geqslant \gamma_u(|u|) \Rightarrow \nabla V(x) f(x,u) \leqslant -\alpha(V(x)),\ \text{a.e.,} \tag{71}$$

where $\gamma \in \mathcal{K}$ and α is a continuous and positive definite function. Then, for any $\sigma \in \mathcal{K}_\infty$ being locally Lipschitz on $(0,\infty), \bar{V} := \sigma(V)$ is also continuously differentiable almost everywhere, and there exists a continuous and positive definite function $\bar{\alpha}$ such that

$$\begin{aligned}&\bar{V}(x) \geqslant \sigma \circ \gamma(|u|) \Rightarrow \\ &\nabla \bar{V}(x) f(x,u) \leqslant -\bar{\alpha}(\bar{V}(x)),\ \text{a.e..}\end{aligned} \tag{72}$$

Such transformation is also valid for discrete-time systems.

Example 2 Consider the continuous-time dynamical network (14). Define $\bar{V}_i = \sigma_i(V_i)$ for $i = 1,2,3$ with $\sigma_i \in \mathcal{K}_\infty$ being locally Lipschitz on $(0,\infty)$. Then, $\bar{V}_i$ is still an ISS-Lyapunov function of the x_i-subsystem and satisfies

$$\begin{aligned}&\bar{V}_i(x_i) \geqslant \max_{j\neq i}\{\bar{\gamma}_{ij}(\bar{V}_i(x_i))\} \Rightarrow \\ &\nabla V_i(x_i) f_i(x) \leqslant -\bar{\alpha}_i(V_i(x_i)),\ \forall x,\end{aligned} \tag{73}$$

where $\bar{\alpha}_i$ is a continuous and positive definite function, and $\bar{\gamma}_{ij} = \sigma_i \circ \gamma_{ij} \circ \sigma_j^{-1}$.

Now we consider the special case where $\gamma_{12}, \gamma_{13}, \gamma_{21}, \gamma_{32}, \gamma_{31}$ are of class $\mathcal{K}_\infty$ and are continuously differentiable on $(0,\infty)$. Also assume that the cyclic-small-gain condition is satisfied, i.e.,

$$\gamma_{13} \circ \gamma_{31} < \mathrm{Id}, \tag{74}$$

$$\gamma_{13} \circ \gamma_{32} \circ \gamma_{21} < \mathrm{Id}. \tag{75}$$

Then, by using Lemma C.1[51], there exists a continuous and positive definite δ such that

$$(\gamma_{13} + \delta) \circ \gamma_{31} < \mathrm{Id}, \tag{76}$$

$$(\gamma_{13} + \delta) \circ \gamma_{32} \circ (\gamma_{21} + \delta) < \mathrm{Id}. \tag{77}$$

By choosing $\sigma_1 = \gamma_{21} + \delta, \sigma_2 = \mathrm{Id}$ and $\sigma_3 = (\gamma_{21} + \delta) \circ (\gamma_{13} + \delta)$, we can show that all the interconnection gains $\bar{\gamma}_{ij}$'s are less than Id, as shown in Figure 6.

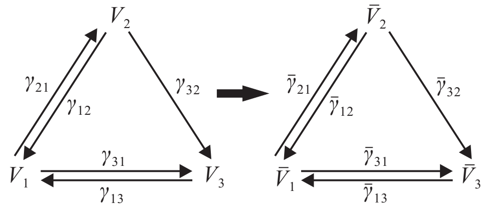

Figure 6　The equivalence between cyclic-small-gain and gains less than Id: $\bar{\gamma}_{ij} = \sigma_i \circ \gamma_{ij} \circ \sigma_j^{-1}$

Under Asuumption 1, the rest of this subsection shows that, if a hybrid dynamical network satisfies the cyclic-small-gain condition, then we can find a transformation $\sigma_i(V_i)$ for each V_i as done in Example 2, such that the interconnection gains in the dynamical network are less than Id.

The first step is to replace the ISS gains $\gamma_{(\cdot)}$'s with $\hat{\gamma}_{(\cdot)}$'s such that

1) $\hat{\gamma}_{(\cdot)} \in \mathcal{K}_\infty$ and $\hat{\gamma}_{(\cdot)} > \gamma_{(\cdot)}$ if $\gamma_{(\cdot)} \in \mathcal{K}; \hat{\gamma}_{(\cdot)} = 0$ if $\gamma_{(\cdot)} = 0$.

2) $\hat{\gamma}_{(\cdot)}$'s are locally Lipschitz on $(0,\infty)$, and

3) For each $r = 2, 3, \cdots, N$,

$$\hat{\gamma}_{i_1 i_2} \circ \cdots \circ \hat{\gamma}_{i_r i_1} < \mathrm{Id} \tag{78}$$

holds for all $1 \leqslant i_j \leqslant N$ and $i_j \neq i_{j'}$ if $j \neq j'$.

The existence of such $\hat{\gamma}_{(\cdot)}$'s can be guaranteed for hybrid dynamical networks, by reasoning similar to continuous-time dynamical networks, see Subsection 2.1.2. The only difference is that, here we consider a more general case in which $\hat{\gamma}_{(\cdot)}$'s are chosen to be locally Lipschitz on $(0, \infty)$.

Recall that $\mathcal{RS}(i)$ represents the reaching set of the i-th subsystem. For a set $\varXi$ satisfying $\bigcup\limits_{i \in \varXi} \mathcal{RS}(i) = \mathcal{N}$, define

$$\varGamma^{[q \to \varXi]}(s) = \bigcup_{p \in \varXi} \bigcup_{j \in \mathcal{N}} \varGamma_j^{[q \to p]}(s), \tag{79}$$

where

$$\varGamma_j^{[q \to p]}(s) = \\ \{\hat{\gamma}_{0 i_1^{[q \to p]}} \circ \hat{\gamma}_{i_1^{[q \to p]} i_2^{[q \to p]}} \circ \cdots \circ \hat{\gamma}_{i_{j-1}^{[q \to p]} i_j^{[q \to p]}}(s)\} \tag{80}$$

for $s \geqslant 0$, with $\hat{\gamma}_{0 i_1^{[q \to p]}} = \mathrm{Id}, i_1^{[q \to p]} = q, i_k^{[q \to p]} \in \mathcal{N}, k \in \{1, 2, \cdots, j\}, i_k^{[q \to p]} \neq i_{k'}^{[q \to p]}$ if $k \neq k'$, for $j \in \mathcal{N}$. Clearly, each $\varGamma_j^{[q \to p]}$ corresponds to a simple path from the x_q-subsystem to the x_p-subsystem, and $\varGamma^{q \to \varXi}$ corresponds to the simple paths from the x_q-subsystem to all the x_p-subsystems with $p \in \varXi$. A special case is $q = p$ and $j = 1$.

Define

$$\begin{aligned} &\bar{\boldsymbol{V}}_q^{\varXi}(x_q) = \varGamma^{[q \to \varXi]}(V_q(x_q)), \\ &\bar{V}_q^{\varXi}(x_q) = \hat{\gamma}^{[q \to \varXi]}(V_q(x_q)) \end{aligned} \tag{81}$$

with $\hat{\gamma}^{[q \to \varXi]}(s) = \max \varGamma^{[q \to \varXi]}(s)$ for $s \geqslant 0$. Clearly, $\bar{V}_q^{\varXi}(x_q) = \max \bar{\boldsymbol{V}}_q^{\varXi}(x_q)$.

As there exists at least one subsystem in $\varXi$ that is reachable from the x_q-subsystem and $\hat{\gamma}_{(\cdot)} \in \mathcal{K}_\infty \bigcup \{0\}$, it can be observed that for all $q \in \mathcal{N}, \hat{\gamma}^{[q \to \varXi]} \in \mathcal{K}_\infty$. Moreover, $\bar{V}_q^{\varXi}$ is positive definite and radially unbounded with respect to x_q, and is continuously differentiable almost everywhere. Correspondingly, to simplify the discussions, we define

$$\bar{u}_q^{\varXi} = \hat{\gamma}^{[q \to \varXi]} \circ \gamma_{u_q}(|u_q|) \tag{82}$$

as the new input of the x_q-subsystem.

Proposition 1 presents the main result on the equivalence between cyclic-small-gain and interconnection gains less than Id.

Proposition 1 A hybrid dynamical network composed of subsystems (66) and (67) which satisfy Assumption 1 and the cyclic-small-gain condition (28) can be reformulated as one with interconnection gains less than Id by considering $\bar{V}_q^{\varXi}$ defined in (81) as the new ISS-Lyapunov function and $\bar{u}_q^{\varXi}$ defined in (82) as the new input for each x_q-subsystem with $q \in \mathcal{N}$.

The proof of Proposition 1 is given in Subsection 3.3.1[51].

2.5.2 Cyclic-small-gain theorem for hybrid dynamical networks

In this subsection, we first consider the dynamical networks composed of subsystems in the form of (66) and (67) with interconnection gains γ_{ij}'s $(i, j \in \mathcal{N},\ i \neq j)$ defined in (68) and (69) less than Id.

For a continuous-time dynamical network composed of two subsystems, if the interconnection gains are less than Id, then one may choose $\sigma = \mathrm{Id}$ in (29), and construct an ISS-Lyapunov function as the maximum of the Lyapunov functions of the subsystems. Similarly, for the hybrid dynamical networks with interconnection gains less than the identity, we construct the following Lyapunov function candidate

$$V(x) = \max \boldsymbol{V}(x) \tag{83}$$

with

$$\boldsymbol{V}(x) = \{V_1(x_1), V_2(x_2), \cdots, V_N(x_N)\}. \tag{84}$$

Since for each $i \in \mathcal{N}, V_i(x_i)$ is positive definite and radially unbounded with respect to x_i, it can be verified that $V(x)$ is positive definite and radially unbounded with respect to x. Moreover, V is locally Lipschitz on $\boldsymbol{R}^n \backslash \{0\}$.

Correspondingly, we define

$$\bar{u} = \max \boldsymbol{U} \tag{85}$$

with

$$\boldsymbol{U} = \{\gamma_{u_1}(|u_1|), \gamma_{u_2}(|u_2|), \cdots, \gamma_{u_N}(|u_N|)\}. \tag{86}$$

Denote $\pi = \bigcup\limits_{i \in \mathcal{N}} \pi_i$ as the set of the impulsive time instants of the dynamical network. The following theorem shows that the hybrid dynamical network with gains less than Id is ISS with $V(x)$ defined in (83) as a weak Lyapunov function in the sense that $V(x(t))$ is, not necessarily strictly, decreasing along the solutions $x(t)$.

Theorem 9 Consider the hybrid dynamical network composed of subsystems (66) and (67). Under Assumption 1, if all the interconnection gains γ_{ij}'s $(i, j \in \mathcal{N},\ i \neq j)$ are less than Id, i.e., $\gamma_{ij} < \mathrm{Id}$, then $V(x)$ defined in (83) is a weak ISS-Lyapunov function and admits the following properties:

1) For any ξ, u and $t_0 \geqslant 0$,

$$V(x(t, t_0, \xi, u)) \geqslant \bar{u}(t) \Rightarrow \dot{V}(x(t, t_0, \xi, u)) \leqslant 0 \tag{87}$$

holds for almost all $t \in [t_0, \infty) \backslash \pi$;

2) For any ξ, u and $t_0 \geqslant 0$,

$$V(x(t, t_0, \xi, u)) \leqslant \\ \max\{V(x(t^-, t_0, \xi, u)), \bar{u}(t^-)\} \tag{88}$$

holds for all $t \in (t_0, \infty) \bigcap \pi$;

3) There exist a $\overline{\delta t}_D > 0$ and a positive definite function ρ^* satisfying $(\mathrm{Id} - \rho^*) \in \mathcal{K}_\infty$, such that for any ξ and any u,

$$\begin{aligned}&V(x(t, t_0, \xi, u)) \leqslant \\ &\max\{(\mathrm{Id} - \rho^*)(V(\xi)), \|\bar{u}\|_{[t_0,t]}\}\end{aligned} \tag{89}$$

holds for any pair of nonnegative numbers (t, t_0) satisfying $t - t_0 \geqslant \overline{\delta t}_D$, and the hybrid dynamical network is ISS.

The proof of Theorem 9 is given in Appendix D.4[51].

Based on the observation that a hybrid dynamical network satisfying the cyclic-small-gain condition can be reformulated as one with interconnection gains less than the identity function, we develop a cyclic-small-gain theorem for hybrid dynamical networks.

If a hybrid dynamical network composed of subsystems (66) and (67) satisfies the cyclic-small-gain condition (28), based on Proposition 1 and Theorem 9, we can construct an ISS-Lyapunov function as

$$V^{\Xi}(x) = \max_{q \in \mathcal{N}}\{\bar{V}_q^{\Xi}(x_q)\}. \tag{90}$$

Corresponding to V^{Ξ}, we define

$$\bar{u}^{\Xi} = \max_{q \in \mathcal{N}}\{\bar{u}_q^{\Xi}\} \tag{91}$$

as the new input of the hybrid dynamical network. Then, properties (87) ~ (89) hold for the hybrid dynamical network with V replaced by V^{Ξ} and $\bar{u}$ replaced by $\bar{u}^{\Xi}$. Our main theorem is as follows:

Theorem 10 A hybrid dynamical network composed of subsystems (66) and (67) satisfying the cyclic-small-gain condition (28) is ISS with V^{Ξ} defined in (90) as a weak ISS-Lyapunov function.

This section has presented cyclic-small-gain results for continuous-time, discrete-time, and hybrid dynamical networks with the ISS property of the subsystems formulated by ISS-Lyapunov functions based on recent results in [27-30, 41]. Further extensions to (hybrid) dynamical networks with time delays deserve future studies, although some preliminary results are available for some classes of large-scale nonlinear systems[22, 42, 51].

3 Control applications

The cyclic-small-gain theorem as mentioned above has been used to solve several challenging control problems of practical importance for nonlinear systems subject to information constraints. Examples of control applications include networked control systems, quantized nonlinear control, event-based nonlinear control and distributed control of multi-agent nonlinear systems. Take the benchmark classes of strict-feedback and output-feedback form nonlinear systems as an example. The basic idea of the corresponding small-gain designs for addressing quantized nonlinear control problems is to transform the closed-loop systems into interconnections of subsystems, and design the interconnection gains to satisfy the cyclic-small-gain condition. It should be noted that nontrivial extensions of the standard designs have been made to handle the new problems caused by information constraints. For instance, due to the discontinuity caused by quantization, the standard backstepping design[54] cannot be readily applied to recursive design of quantized controllers for nonlinear systems in the lower-triangular form. To solve this problem, a new design method based on set-valued maps is developed such that the closed-loop quantized system can be transformed into an interconnection of ISS subsystems, and the quantized control objective is achieved by appropriately tuning the gains of the subsystems[55-57].

Other related applications of the nonlinear small-gain method can be found in a series of our recent work in robust measurement feedback control (with sensor noise), decentralized control, distributed control and event-triggered control. Interested readers may consult references [51, 57-63].

4 Conclusions

The idea of small-gain is about 40 years old. Along with thinking of the Nyquist criterion interpretation, it is one of the two most fundamental ideas for preserving stability in a feedback loop. Nevertheless, after all the work of numerous researchers, taking this idea to ever more complex dynamical controlled networks still has a long way to go. The topics considered in this paper show that many interesting research problems, in particular at the level of control synthesis, remain to be studied. These new challenges relate to theoretical advances as well as further application of the small-gain results. Some suggestions for such future work are listed as follows:

1) Small-gain results for large-scale hybrid systems which may involve both stable dynamics and unstable dynamics have not been addressed in greater details, and there seem to be possibilities to advance the complexity to include switching, impulses, delays, and interconnections.

2) Despite the popularity of the standard feedback forms of nonlinear systems, developing new design

tools for nonlinear MIMO (multi-input multi-output) systems with more general structures would contribute tremendously to the development of a complete theory of modern nonlinear control.

3) Although considerable efforts have been made on the analysis of networked nonlinear control systems, it is still highly desirable to refine the standard constructive design tools by taking into account network-induced issues such as communication delays, loss of communication packets, and impact of networking protocols on the performance of the closed-loop network system. Cyber-physical systems is a largely unexplored research area where network small-gain theory as reviewed in this article can play a vital role in terms of both system analysis and control synthesis.

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