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基于易腐品和不可分割耐用品的最优消费投资与寿险购买策略

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摘 要: 随着经济的发展和人民生活水平的提高,投资者的投资组合不再局限于证券市场投资. 通过将寿险购买引入投资者的投资组合并划分消费品为易腐品或不可分割耐用品,研究投资者的最优消费投资与寿险购买策略. 投资者的投资目标为期望效用最大化. 运用动态规划原理得到哈密尔顿-雅可比-贝尔曼方程,最终得到最优策略满足的方程,并讨论方程存在正根的条件. 最后通过数值分析方法,验证模型结论与实际现实情况的一致性.

关键词: 最优消费投资与寿险购买策略；不可分割耐用品；动态规划；随机控制

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Optimal consumption, investment and insurance purchase strategies based on perishable and indivisible durable consumption good

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Abstract: With the development of economy and the improvement of people's living standard, the investor's investment is no longer limited to the portfolio of the financial assets. This paper studies optimal consumption, investment and insurance purchase strategies by introducing the insurance purchase into the portfolio of the investor and dividing consumption into the perishable good or the indivisible durable good. The investor's objective is the maximum of expected utility. By using the theory of dynamic programming, the Hamilton-Jacobi-Bellman equation is obtained, the equation of existence of the optimal strategy is derived, and the condition of the positive root satisfying the equation is discussed. Finally a numerial analysis verifies that the conclusion of the proposed model fits the empirical behavior in reality.

Keywords: optimal consumption-investment-insurance purchase strategy；indivisible durable good；dynamic programming；stochastic control

0 引 言

自Merton[1]提出投资消费理论以来,数理金融成为了各国学者研究的热点. 基于Merton[1],大量成果涌现出来,使得投资消费理论渐趋完善,具体可归纳为以下几个方面: 1) 在模型中采用不同的效用函数求取最优策略,如Song等[2]采用S型效用函数求取了具有下行消费约束的损失厌恶投资者在无限时域条件下的最优消费投资策略,并与CRRA效用函数的投资者最优消费投资策略进行了对比;Chang等[3]采用HARA效用函数(HARA效用函数包含CRRA、CARA和对数效用)求取投资者的最优投资消费策略. 2) 将随机利率引入模型之中,如Vasicek[4]提出Vasicek随机利率模型之后,许多学者将其引入到无风险资产利率的设定中,求取在随机利率条件下投资者的最优消费投资策略;Chang等[3]在模型中引入Vasicek随机利率获得投资者的最优消费投资策略. 3) 在模型中引入随机波动率,如Wang等[5]研究了在随机波动率条件下的脆弱期权定价问题. 4) 在模型的设定中同时引入随机利率与随机波动率,如Chang[6]所提模型. 5) 采用不同的数学方法求取最优策略,如Kwak等[7]

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求解模型时采用的是鞅方法; Chang等[3]采用动态规划原理, 得到哈密尔顿-雅可比-贝尔曼方程(HJB方程), 最终得到投资者的最优消费投资策略. 6) 模型中存在借贷约束或消费约束, 如Song等[2]和Ma等[8]所提模型. 7) 存在交易费用, 如Liu等[9]研究了小比例交易费用条件下的资产配置问题; Frutos等[10]研究了带有交易费用的有限时域最优投资问题的数值方法问题.

随着经济社会的发展, 保险市场应运而生, 投资者的投资品种不再仅仅涉及到金融市场上的证券投资, 即投资无风险资产(如国债等)与风险资产(如股票等), 越来越多的投资者选择购买寿险来规避风险. 根据我国保险监督管理委员会公布的数据: 2017年1～11月, 寿险公司原保险保费收入24 914.21亿元, 同比增长22.11 %; 寿险业务原保险保费收入17 442.22亿元, 同比增长31.72 %. 然而, 当寿险进入投资者的投资组合后, 投资者的最优策略势必会发生变化, 因而近些年含有寿险的最优策略成为国内外学者关注的热点. Pliska等[11]引入单一风险资产得到了最优投资消费与寿险购买策略; 之后, Duarte等[12]采用n种风险资产(n是任意但有限的)得到了投资者的最优消费投资与寿险购买策略; Kwak等[7]则研究了通胀风险下投资者的最优消费投资与寿险购买策略问题; Han等[13]解决了在随机利率与通胀风险并存条件下的最优消费投资与寿险购买问题.

另外, 近年来世界人口爆发式增长, 由于资源的有限性, 像汽车和房地产等耐用品也受到了投资者的青睐. Domenico[14]研究了包含可分割耐用品的最优投资消费策略; Anders[15]将消费品划分为易腐品与不可分割的耐用品, 得到了在不存在交易成本与存在交易成本两种情况下投资者的最优消费投资策略.

本文将寿险与易腐品和耐用品同时引入组合投资消费模型之中, 这样与现实中投资者面临的投资环境更为贴近. 假定消费者购买易腐品与耐用品以及最终财富均可给投资者带来效用, 最终得到满足最优消费投资与寿险购买策略的极为复杂的方程. 通过分类讨论思想, 求取存在解的条件, 并给出最优策略的表达式, 使得可通过数值分析方法得到投资者的最优策略. 最后, 通过数值分析方法验证模型符合现实条件下投资者的投资行为.

1 模型建立

定义一个概率空间$(\Omega,\mathcal{F},P)$, $\mathcal{F}$是域流. 首先, 假定金融市场上存在$n+1$种金融资产: 一个无风险资产(如国债等), n个风险资产(如股票等). 无风险资产的价格过程为

$$\mathrm{d}p_0(t)=r(t)p_0(t)\mathrm{d}t, \tag{1}$$

其中$r(t)$是t时刻的无风险利率, 且$r(t)\in R^+$.

风险资产的价格过程为

$$\mathrm{d}p(t)=\mathrm{diag}(p(t))[b(t)\mathrm{d}t+\sigma(t)\mathrm{d}W^1(t)]. \tag{2}$$

其中: $p(0)=p$; $p(t)$是n维价格向量; $\mathrm{diag}(p(t))$是一个方阵, 主对角线元素由向量$p(t)$的各元素构成, 其余元素为0; $b(t)$是瞬时期望收益率, 且$b(t)\in\boldsymbol{R}^n$; $\sigma(t)$是$n\times n$维的非奇异矩阵; $W^1(t)$是定义在$(\Omega,\mathcal{F},P)$上的n维维纳过程, 从而风险资产的价格过程是相互关联的几何布朗运动.

然后, 考虑投资者面临的消费市场问题. 细分消费市场中的消费品为易腐品(不能存储, 且只能在购买时提供效用, 例如食品等)和耐用品(可贮藏, 可出售, 且在一段时间内均可为投资者提供效用, 例如汽车、房产等), 且消费市场中的消费品要么是易腐品要么是耐用品. 假定易腐品的消费率为$c(t)$, 耐用品的价格过程为$s(t)$, 并且满足

$$\begin{aligned}&\mathrm{d}s(t)=\\&s(t)[\mu(t)\mathrm{d}t+\sigma_1(t)^{\mathrm{T}}\mathrm{d}W^1(t)+\sigma_2(t)\mathrm{d}W^2(t)].\end{aligned} \tag{3}$$

其中: $s(0)=s$; $\mu(t)$和$\sigma_2(t)$是常量; $(\cdot)^{\mathrm{T}}$表示向量或矩阵的转置形式; $\sigma_1(t)\in\boldsymbol{R}^n$是一个常数向量; $W^2(t)$是定义在$(\Omega,\mathcal{F},P)$上的一维维纳过程; $W^2(t)$与$W^1(t)$不相关. 为了后续计算的简便, 定义$n+1$维向量$\sigma_s(t)=(\sigma_1^{\mathrm{T}}(t),\sigma_2(t))^{\mathrm{T}}$. 通过这种构建方式, 金融资产的价格过程与耐用品的价格过程不完全相关, 因此不能通过在金融市场合理的投资行为来对冲耐用品价格的所有风险.

考虑到耐用品存在一个物理折旧过程, 设其折旧率为δ. 在t时刻投资者持有的耐用品数量为$M(t)$, 且满足

$$\mathrm{d}M(t)=-\delta M(t)\mathrm{d}t, \tag{4}$$

最后, 考虑投资者面临的保险市场. 根据精算学理论, 定义τ为一个非负随机变量, 代表投资者的死亡时间. 定义$\lambda(t)$为风险函数或投资者的瞬时死亡率, 且满足

$$\lambda(t)=-\frac{\mathrm{d}}{\mathrm{d}t}(\ln F(t)), \tag{5}$$

其中$F(t)$为生存函数, 且

$$F(t)=\exp\Big(-\int_0^t\lambda(u)\mathrm{d}u\Big). \tag{6}$$

$f(t)$为随机变量τ的概率密度函数, 且

$$f(t)=\lambda(t)\exp\Big(-\int_0^t\lambda(u)\mathrm{d}u\Big). \tag{7}$$

从而可知投资者在$g\geqslant t$生存, 在g时刻死亡的条件

概率密度为

$$f(g,t)=\frac{f(g)}{F(t)}=\lambda(g)\exp\Big(-\int_t^g\lambda(u)\mathrm{d}u\Big). \tag{8}$$

投资者在$g\geqslant t$生存,在g时刻死亡的条件概率为

$$\overline{F}(g,t)=\exp\Big(-\int_t^g\lambda(u)\mathrm{d}u\Big). \tag{9}$$

与Pliska等[11]的研究相一致,假定投资者在t时刻保费投入为$\bar{p}(t)$,若投资者在$t(0\leqslant t<T\bigwedge\tau)$时刻死亡,保险公司将赔付$\bar{p}(t)/\eta(t)$给受益人. 其中:$T$是投资者退休时间,$T\bigwedge\tau=\min\{T,\tau\}$,$\eta(t)$是保险赔付率. 为了确保保险公司的利润不得为负,要求$\eta(t)\geqslant\lambda(t)$. 显而易见,若被保险人的死亡时间为$t(t>T)$,则受益人无法得到赔付.

定义在t时刻投资者投资于无风险资产的金额为$\pi_0(t)$;$\pi_i(t)$为在t时刻投资于第i种$(i\in N_+)$风险资产的金额,且$\pi(t)=(\pi_1(t),\cdots,\pi_n(t))^{\mathrm{T}}$;$\theta(t)=M(t)s(t)$为$t$时刻投资于耐用品的金额. 假定消费品市场不存在交易成本,金融市场上不存在套利行为,初始财富为$x(0)=x$的投资者财富过程为

$$\begin{aligned}&x(t)=\\&\pi_0(t)+\pi(t)^{\mathrm{T}}\mathbf{1}+\theta(t)=\\&N_0(t)P_0(t)+N_1(t)P(t)+M(t)s(t).\end{aligned} \tag{10}$$

其中:$N_0(t)$和$N_1(t)$分别是在t时刻持有的无风险资产和风险资产的数量;$\mathbf{1}=(1,\cdots,1)^{\mathrm{T}}$,是一个$n\times1$维向量.

因此,财富过程的微分形式可表示为

$$\begin{aligned}\mathrm{d}x(t)=&[x(t)r(t)+\pi(t)^{\mathrm{T}}(b(t)-r(t)\mathbf{1})+\\&\theta(t)(\mu(t)-r(t)-\delta)-\bar{p}(t)-c(t)]\mathrm{d}t+\\&(\pi(t)\sigma(t)+\theta(t)\sigma_1(t)^{\mathrm{T}})\mathrm{d}W^1(t)+\\&\theta(t)\sigma_2(t)\mathrm{d}W^2(t).\end{aligned} \tag{11}$$

当投资者在$t\,(0<t<T)$时刻死亡时,由于保险公司的赔付,其总遗产为

$$Z(t)=x(t)+\frac{\bar{p}(t)}{\eta(t)}. \tag{12}$$

故对投资者的允许策略$(c,\bar{p},\theta,\pi)\in\varLambda(x,s)$所满足的期望效用函数形式为

$$\begin{aligned}J(x,s)=E\Big\{&\int_t^{T\wedge\tau}[f(u,t)\mathrm{e}^{-\rho u}U_2(Z(u))+\\&\overline{F}(u,t)\mathrm{e}^{-\rho u}U_1(c(u),\hat{\theta}(t))]\mathrm{d}u+\\&\overline{F}(T,t)\mathrm{e}^{-\rho T}U_2(x(T))\Big\}.\end{aligned} \tag{13}$$

其中:$f(u,t)$由式(8)给定;$\overline{F}(u,t)$由式(9)给定;$\hat{\theta}(t)=\theta(t)/s(t)=M(t)$;$\rho>0$是贴现率;$U_1(\cdot)$和$U_2(\cdot)$是效用函数.

根据文献[11],可得式(13)的等价形式

$$\begin{aligned}J(x,s)=E\Big\{&\int_t^{\mathrm{T}}[f(u,t)\mathrm{e}^{-\rho u}U_2(Z(u))+\\&\overline{F}(u,t)\mathrm{e}^{-\rho u}U_1(c(u),\hat{\theta}(t))]\mathrm{d}u+\\&\overline{F}(T,t)\mathrm{e}^{-\rho T}U_2(x(T))\Big\}.\end{aligned} \tag{14}$$

投资者的目标在于最大化期望效用,故可得值函数的形式为

$$V(x,s)=\sup_{(c,\pi,\bar{p},\theta)\in\varLambda(x,s)}J(x,s). \tag{15}$$

对于$0\leqslant t<t'<T$,最大化期望效用函数满足以下递推关系式:

$$\begin{aligned}&V(x,s)=\\&\sup_{(c,\pi,\bar{p},\theta)\in\varLambda(x,s)}\Big\{E\Big[\int_t^{t'}[f(u,t)\mathrm{e}^{-\rho u}U_2(Z(u))+\\&\overline{F}(u,t)\mathrm{e}^{-\rho u}U_1(c(u),\hat{\theta}(t))]\mathrm{d}u+\\&\mathrm{e}^{-\int_t^{t'}\lambda(v)\mathrm{d}v}V(x+\Delta x,s+\Delta s)|\mathcal{F}_t\Big]\Big\}.\end{aligned} \tag{16}$$

由动态规划原理可得

$$\begin{aligned}&\lambda(t)V=\\&\sup\Big\{\lambda(t)\mathrm{e}^{-\rho t}U_2\Big(x+\frac{\bar{p}(t)}{\eta(t)}\Big)+\mathrm{e}^{-\rho t}\times\\&U_1(c(t),\hat{\theta}(t))+V_x[x(t)r(t)+\pi(t)^{\mathrm{T}}(b(t)-\\&r(t)\mathbf{1})+\theta(t)(\mu(t)-r(t)-\delta)-\bar{p}(t)-c(t)]+\\&V_s\mu(t)s(t)+s(t)V_{xs}(\pi(t)^{\mathrm{T}}\sigma(t)\sigma_1(t)+\\&\theta(t)\|\sigma_s(t)\|^2)+\frac{1}{2}V_{xx}(\pi(t)^{\mathrm{T}}\sigma(t)\sigma(t)^{\mathrm{T}}\pi(t)+\\&\theta(t)^2\|\sigma_s(t)\|^2+2\pi(t)^{\mathrm{T}}\theta(t)\sigma(t)\sigma_1(t))+\\&\frac{1}{2}V_{ss}s(t)^2\|\sigma_s(t)\|^2\Big\}.\end{aligned} \tag{17}$$

其中

$$V_x=\frac{\partial V(x,s)}{\partial x},\ V_{xx}=\frac{\partial^2V(x,s)}{\partial x^2},$$
$$V_s=\frac{\partial V(x,s)}{\partial s},\ V_{ss}=\frac{\partial^2V(x,s)}{\partial s^2},$$
$$V_{xs}=\frac{\partial^2V(x,s)}{\partial x\partial s},$$

$\|\cdot\|$是向量或矩阵的模.

假设投资者的效用函数是等弹性可分的形式,即

$$U_1(c(t),\hat{\theta}(t))=\frac{1}{1-\gamma}(c(t)^{\beta}\hat{\theta}(t)^{1-\beta})^{1-\gamma}, \tag{18}$$

$$U_2\Big(x+\frac{\bar{p}(t)}{\eta(t)}\Big)=\frac{1}{1-\gamma}\Big(x+\frac{\bar{p}(t)}{\eta(t)}\Big)^{1-\gamma}, \tag{19}$$

其中$\beta,\gamma\in(0,1)$.

令$\hat{\pi}(t)=\dfrac{\pi(t)}{s(t)}$,$\theta\hat{(t)}=\dfrac{\theta(t)}{s(t)}$,$\hat{c}(t)=\dfrac{c(t)}{s(t)}$,$y(t)=\dfrac{x(t)}{s(t)}$,$\hat{p}(t)=\dfrac{\bar{p}(t)}{s(t)}$,并猜想$V(x,s)=s^{\beta(1-\gamma)}V(x/s,1)$

$= s^{\beta(1-\gamma)}V_1(x/s)$,将其代入式(17)可得

$$\Big\{\lambda(t)-\beta(1-\gamma)\Big[\mu(t)-\frac{1}{2}(1-\beta(1-\gamma))\times\|\sigma_s(t)\|^2\Big]\Big\}V_1(y)=\sup\Big\{\lambda(t)\mathrm{e}^{-\rho t}\frac{1}{1-\gamma}\Big(y+\frac{\hat{p}(t)}{\eta(t)}\Big)^{1-\gamma}s^{(1-\gamma)(1-\beta)}+V_1{}'(y)[(r(t)+(1-\beta(1-\gamma))\|\sigma_s(t)\|^2-\mu(t))\times(y(t)-\hat{\theta}(t))-\delta\hat{\theta}(t)+\hat{\pi}(t)^{\mathrm{T}}(b(t)-r(t)\mathbf{1}-(1-\beta(1-\gamma))\sigma(t)\sigma_1(t))-\hat{p}(t)-\hat{c}(t)]+\frac{1}{2}V_1{}''(y)\times[\hat{\pi}(t)^{\mathrm{T}}\sigma(t)\sigma(t)^{\mathrm{T}}\hat{\pi}(t)+\|\sigma_s(t)\|^2(y(t)-\hat{\theta}(t))^2-2\hat{\pi}(t)^{\mathrm{T}}\sigma(t)\sigma_1(t)(y(t)-\hat{\theta}(t))]+\mathrm{e}^{-\rho t}\frac{1}{1-\gamma}(\hat{c}(t)^{\beta}\hat{\theta}(t)^{1-\beta})^{1-\gamma}\Big\}. \tag{20}$$

假设$V_1(y)=\dfrac{1}{1-\gamma}a_vy^{1-\gamma}$,代入式(20)整理可得

$$\Big\{\lambda(t)-\beta(1-\gamma)\Big[\mu(t)-\frac{1}{2}(1-\beta(1-\gamma))\|\sigma_s(t)\|^2\Big]\Big\}\frac{1}{1-\gamma}a_vy^{1-\gamma}=\sup f(\hat{\pi}(t),\hat{\theta}(t),\hat{c}(t),\hat{p}(t))=\sup\Big\{\lambda(t)\mathrm{e}^{-\rho t}\frac{1}{1-\gamma}\Big(y+\frac{\hat{p}(t)}{\eta(t)}\Big)^{1-\gamma}s^{(1-\gamma)(1-\beta)}+a_v\times y^{-\gamma}\big[(r(t)+(1-\beta(1-\gamma))\|\sigma_s(t)\|^2-\mu(t))\times(y(t)-\hat{\theta}(t))-\delta\times\hat{\theta}(t)+\hat{\pi}(t)^{\mathrm{T}}(b(t)-r(t)\mathbf{1}-(1-\beta(1-\gamma))\sigma_t\sigma_{1t})-\hat{p}_t-\hat{c}_t\big]-[\hat{\pi}(t)^{\mathrm{T}}\sigma(t)\sigma(t)^{\mathrm{T}}\hat{\pi}(t)+\|\sigma_s(t)\|^2(y(t)-\hat{\theta}(t))^2-2\hat{\pi}(t)\sigma(t)\sigma_1(t)(y(t)-\hat{\theta}(t))]\times\frac{1}{2}\gamma a_vy^{-\gamma-1}+\mathrm{e}^{-\rho t}\frac{1}{1-\gamma}(\hat{c}_t^{\beta}\hat{\theta}_t^{1-\beta})^{1-\gamma}\Big\}. \tag{21}$$

式(21)的一阶条件为

$$\frac{\partial f}{\partial\hat{\pi}(t)}=a_vy^{-\gamma}[b(t)-r(t)1-(1-\beta(1-\gamma))\times\sigma(t)\sigma_1(t)]-\gamma a_vy^{-\gamma-1}[\sigma(t)\sigma(t)^{\mathrm{T}}\hat{\pi}(t)^{\mathrm{T}}-\sigma(t)\sigma_1(t)(y(t)-\hat{\theta}(t))]=0, \tag{22}$$

$$\frac{\partial f}{\partial\hat{\theta}(t)}=\mathrm{e}^{-\rho t}(1-\beta)\hat{c}(t)^{\beta(1-\gamma)}\hat{\theta}(t)^{-\gamma-\beta(1-\gamma)}-a_vy^{-\gamma}[r(t)+(1-\beta(1-\gamma))\|\sigma_s(t)\|^2-\mu(t)+\delta]+\gamma a_vy^{-\gamma-1}[\|\sigma_s(t)\|^2\times(y(t)-\hat{\theta}(t))-\hat{\pi}(t)^{\mathrm{T}}\sigma(t)\sigma_1(t)]=0, \tag{23}$$

$$\frac{\partial f}{\partial\hat{c}(t)}=\mathrm{e}^{-\rho t}\beta\hat{c}(t)^{\beta(1-\gamma)-1}\hat{\theta}(t)^{(1-\beta)(1-\gamma)}-a_vy^{-\gamma}=0, \tag{24}$$

$$\frac{\partial f}{\partial\hat{p}(t)}=\lambda(t)\mathrm{e}^{-\rho t}\Big(y+\frac{\hat{p}(t)}{\eta(t)}\Big)^{-\gamma}\frac{1}{\eta(t)}s^{(1-\gamma)(1-\beta)}-a_vy^{-\gamma}=0. \tag{25}$$

由式(24)、$\hat{\theta}(t)=a_\theta y$和$\hat{c}(t)=a_cy$可得

$$a_v=\mathrm{e}^{-\rho t}\beta a_c^{\beta(1-\gamma)-1}a_\theta^{(1-\beta)(1-\gamma)}. \tag{26}$$

由式(25)和$\hat{p}(t)=a_py$可得

$$a_v=\lambda(t)\mathrm{e}^{-\rho t}\Big(1+\frac{a_p}{\eta(t)}\Big)^{-\gamma}\frac{1}{\eta(t)}s^{(1-\beta)(1-\gamma)}. \tag{27}$$

由式(22)、$\hat{\pi}(t)=a_\pi y$和$\hat{\theta}(t)=a_\theta y$可得

$$a_\pi=\frac{1}{\gamma}(\sigma(t)\sigma(t)^{\mathrm{T}})^{-1}(b(t)-r(t)\mathbf{1}-(1-\gamma)\times(1-\beta)\sigma(t)\sigma_1(t))-(\sigma(t)^{\mathrm{T}})^{-1}\sigma_1(t)a_\theta. \tag{28}$$

由式(23)、$\hat{\pi}(t)=a_\pi y$、$\hat{\theta}(t)=a_\theta y$和$\hat{c}(t)=a_cy$可得

$$\mathrm{e}^{-\rho t}(1-\beta)a_c^{-\gamma-\beta(1-\gamma)}-a_v[r(t)-\mu(t)+\delta+(1-\beta(1-\gamma))]+\gamma a_v[\|\sigma_s(t)\|^2(1-a_\theta)-a_\pi\sigma(t)\sigma_1(t)]=0. \tag{29}$$

将式(28)代入(29),并利用(26)可得

$$a_c=\frac{\beta}{1-\beta}\big[r(t)+(1-\gamma)(1-\beta)\sigma_2(t)^2-\mu(t)+\delta+(b(t)-r(t)\mathbf{1})^{\mathrm{T}}(\sigma(t)^{\mathrm{T}})^{-1}\sigma_1(t)\big]a_\theta+\frac{\beta\gamma}{1-\beta}\sigma_2(t)^2a_\theta^2. \tag{30}$$

由式(26)和(27)可得

$$a_p=\eta(t)\Big[\Big(\frac{\beta}{\lambda(t)}\eta(t)s^{-(1-\gamma)(1-\beta)}\Big)^{-\frac{1}{\gamma}}\times\Big(\frac{\beta\gamma}{1-\beta}\sigma_2(t)^2a_\theta+\frac{\beta}{1-\beta}\Big(r(t)-\mu(t)+\delta+(1-\gamma)(1-\beta)\sigma_2(t)^2+(b(t)-r(t)\mathbf{1})^{\mathrm{T}}(\sigma(t)^{\mathrm{T}})^{-1}\sigma_1(t)\Big)\Big)^{\frac{1-\beta(1-\gamma)}{\gamma}}a_\theta-1\Big]. \tag{31}$$

下面对式(31)进行证明.

证明 由式(26)和(27),有

$$a_p=\eta(t)\Big[\Big(\frac{\beta}{\lambda(t)}\eta(t)s^{-(1-\gamma)(1-\beta)}a_c^{\beta(1-\gamma)-1}\times a_\theta^{(1-\beta)(1-\gamma)}\Big)^{-\frac{1}{\gamma}}-1\Big]=\eta(t)\Big[\Big(\frac{\beta}{\lambda(t)}\eta(t)s^{-(1-\gamma)(1-\beta)}\Big(\frac{\beta\gamma}{1-\beta}\sigma_2(t)^2a_\theta^2+$$

$$\frac{\beta}{1-\beta}((1-\gamma)(1-\beta)\sigma_2(t)^2+r(t)-\mu(t)+\delta+ (b(t)-r(t)\mathbf{1})^{\mathrm{T}}(\sigma(t)^{\mathrm{T}})^{-1}\sigma_1(t))a_\theta\Big)^{\beta(1-\gamma)-1}\times a_\theta^{(1-\beta)(1-\gamma)}\Big)^{-1/\gamma}-1\Big]=$$

$$\eta(t)\Big[\Big(\frac{\beta}{\lambda(t)}\eta(t)s^{-(1-\gamma)(1-\beta)}\Big)^{-\frac{1}{\gamma}}\times \Big(\frac{\beta\gamma}{1-\beta}\sigma_2(t)^2a_\theta+\frac{\beta}{1-\beta}\Big(r(t)-\mu(t)+\delta+ (1-\gamma)(1-\beta)\sigma_2(t)^2+(b(t)- r(t)\mathbf{1})^{\mathrm{T}}(\sigma(t)^{\mathrm{T}})^{-1}\sigma_1(t)\Big)\Big)^{\frac{1-\beta(1-\gamma)}{\gamma}}a_\theta-1\Big].$$

证明成立. □

将$\hat{\pi}(t)=a_\pi y(t),\hat{\theta}(t)=a_\theta y(t),\hat{c}(t)=a_c y(t)$, $\hat{p}(t)=a_p y(t)$和式(26)、(28)、(30)、(31)代入(21)得

$$\frac{\lambda(t)}{1-\gamma}-\beta\Big[\mu(t)-\frac{1}{2}(1-\beta(1-\gamma))\|\sigma_s(t)\|^2\Big]= \frac{\eta(t)}{1-\gamma}\Big(1+\frac{a_p}{\eta(t)}\Big)-a_p+\Big[\frac{1}{\beta(1-\gamma)}-1\Big]a_c+ [r(t)+(1-\beta(1-\gamma))\|\sigma_s(t)\|^2-\mu(t)](1-a_\theta)- \delta a_\theta+a_\pi^{\mathrm{T}}[b(t)-r(t)\mathbf{1}-(1-\beta(1-\gamma))\sigma(t)\sigma_1(t)]- \frac{\gamma}{2}[a_\pi^{\mathrm{T}}\sigma(t)\sigma(t)^{\mathrm{T}}a_\pi+\|\sigma_s(t)\|^2(1-a_\theta)^2- 2a_\pi^{\mathrm{T}}\sigma(t)\sigma_1(t)(1-a_\theta)]. \tag{32}$$

整理式(32)得

$$A_0a_\theta^2+A_1a_\theta+A_2=W(a_p)=B(a_\theta). \tag{33}$$

其中

$$A_0=\Big(\frac{\beta\gamma}{1-\beta}+\frac{1+\gamma}{2}\Big)\sigma_2(t)^2, \tag{34}$$

$$A_1= \frac{1}{1-\beta}\big[r(t)+(1-\gamma)(1-\beta)\sigma_2(t)^2-\mu_t+ \delta+(b(t)-r(t)\mathbf{1})^{\mathrm{T}}(\sigma(t)^{\mathrm{T}})^{-1}\sigma_1(t), \tag{35}$$

$$A_2= -\frac{\rho}{\gamma}+\frac{1-\gamma}{\gamma}\Big\{r(t)-(1-\beta)\mu(t)+\frac{1}{2}(1- \beta)\times[1+(1-\beta)(1-\gamma)]\|\sigma_s(t)\|^2+ \frac{1}{2\gamma}[b(t)-r(t)\mathbf{1}-(1-\beta)(1-\gamma)\sigma(t)\sigma_1(t)]^{\mathrm{T}}- (\sigma(t)\sigma(t)^{\mathrm{T}})^{-1}[b(t)-r(t)\mathbf{1}-(1- \beta)(1-\gamma)\times\sigma(t)\sigma_1(t)]\Big\}, \tag{36}$$

$$W(a_p)=-\Big(\frac{\eta(t)}{1-\gamma}+\frac{\gamma}{1-\gamma}a_p\Big), \tag{37}$$

$$B(a_\theta)= -\eta(t)-\frac{\gamma\eta(t)}{1-\gamma}\Big[\Big(\frac{\beta}{\lambda(t)}\eta(t)s^{-(1-\gamma)(1-\beta)}\Big)^{-\frac{1}{\gamma}}\times \Big(\frac{\beta\gamma}{1-\beta}\sigma_2(t)^2a_\theta+\frac{\beta}{1-\beta}(r(t)-\mu(t)+ \delta+(1-\gamma)\times(1-\beta)\sigma_{2t}^2+(b_t- r_t\mathbf{1})^{\mathrm{T}}(\sigma(t)^{\mathrm{T}})^{-1}\sigma_1(t))\Big)^{\frac{1-\beta(1-\gamma)}{\gamma}}\times a_\theta\Big]. \tag{38}$$

2 最优策略

由式(30)、(34)～(36)得

$$a_c=\frac{\beta\gamma}{1-\beta}\sigma_2(t)^2a_\theta^2+\frac{\beta}{1-\beta}[r(t)-\mu(t)+ \delta+(1-\gamma)(1-\beta)\sigma_2(t)^2+ (b(t)-r(t)\mathbf{1})^{\mathrm{T}}(\sigma(t)^{\mathrm{T}})^{-1}\sigma_1(t)]a_\theta= -\beta A_2-\frac{1}{2}\beta(1-\gamma)\sigma_2(t)^2a_\theta^2. \tag{39}$$

令$A(a_\theta)=A_0a_\theta^2+A_1a_\theta+A_2$,则$A(0)=A_2$. 由$\beta,\gamma\in(0,1)$可知

$$A_0=\Big(\frac{\beta\gamma}{1-\beta}+\frac{1+\gamma}{2}\Big)\sigma_2(t)^2>0.$$

函数的对称轴为$a_\theta=-\dfrac{A_1}{2A_0}$. 由假设$a_c>0,a_p>0,a_\theta>0$可知

$$a_c>0\Rightarrow-\beta A_2-\frac{1}{2}\beta(1-\gamma)\sigma_2(t)^2a_\theta^2>0\Rightarrow$$

$$A_2<-\frac{1}{2}\beta(1-\gamma)\sigma_2(t)^2a_\theta^2<0. \tag{40}$$

令

$$B(a_\theta)=B_0(B_1a_\theta+B_2)^{\frac{1-\beta(1-\gamma)}{\gamma}}a_\theta-\eta(t).$$

其中

$$B_0=-\frac{\gamma\eta(t)}{1-\gamma}\Big(\frac{\beta}{\lambda(t)}\eta(t)s^{-(1-\gamma)(1-\beta)}\Big)^{-\frac{1}{\gamma}}, \tag{41}$$

$$B_1=\frac{\beta\gamma}{1-\beta}\sigma_2(t)^2, \tag{42}$$

$$B_2= \frac{\beta}{1-\beta}\big(r(t)+(1-\gamma)(1-\beta)\sigma_2(t)^2- \mu(t)+\delta+(b(t)-r(t)\mathbf{1})^{\mathrm{T}}(\sigma(t)^{\mathrm{T}})^{-1}\sigma_1(t)\big). \tag{43}$$

因此,$A_1=\beta B_2$.对$B(a_\theta)$求一阶导得

$$B'(a_\theta)= B_0\Big[\frac{1-\beta(1-\gamma)}{\gamma}(B_1a_\theta+B_2)^{\frac{1-\beta(1-\gamma)}{\gamma}-1}\times B_1a_\theta+(B_1a_\theta+B_2)^{\frac{1-\beta(1-\gamma)}{\gamma}}\Big]= B_0(B_1a_\theta+B_2)^{\frac{1-\beta(1-\gamma)}{\gamma}}\Big[\frac{1-\beta(1-\gamma)}{\gamma}\times \frac{1}{B_1a_\theta+B_2}B_1a_\theta+1\Big]. \tag{44}$$

1) 当$A_1 \geqslant 0$时:

$A(a_\theta)$的对称轴$a_\theta = -\dfrac{A_1}{2A_0} \leqslant 0$,则对于$a_\theta > 0$, $A(a_\theta)$单调递增且$B_2 = \dfrac{A_1}{\beta} \geqslant 0, B'(a_\theta) < 0$,故$B(a_\theta)$单调递减. 当且仅当$A(0) \leqslant \lim\limits_{a_\theta \to 0} B(0)$,即$A_2 \leqslant -\eta_t$,方程(33)有且只有唯一正根.

2) 当$A_1 < 0$时:

$A(a_\theta)$的对称轴$a_\theta = -\dfrac{A_1}{2A_0} > 0, B_2 = \dfrac{A_1}{\beta} < 0$. 由$1 - \beta(1-\gamma) - \gamma = (1-\beta)(1-\gamma) > 0$可知, $\dfrac{1-\beta(1-\gamma)}{\gamma} > 1$.

i) 当$\dfrac{1-\beta(1-\gamma)}{\gamma} = \dfrac{j}{2k}(j > 2;\ j \in N_+;\ k = 1, 2, \cdots;\ j$与$2k$互质)时:

$B(a_\theta)$的定义域变为$\Big(-\dfrac{B_2}{B_1}, \infty\Big)$, $\forall a_\theta \in \Big(-\dfrac{B_2}{B_1}, \infty\Big)$, $B'(a_\theta) < 0$恒成立, $B(a_\theta)$在$\Big(-\dfrac{B_2}{B_1}, \infty\Big)$上是单调递减函数. 故当且仅当$A\Big(-\dfrac{B_2}{B_1}\Big) < \lim\limits_{a_\theta \to -\frac{B_2}{B_1}} B(a_\theta) = -\eta_t$时,方程(33)存在唯一正根$a_\theta^*$.

ii) 当$\dfrac{1-\beta(1-\gamma)}{\gamma} = 2n(n = 1, 2, \cdots)$或$\dfrac{1-\beta(1-\gamma)}{\gamma} = \dfrac{j}{2k+1}(j > 2;\ j$为偶数$;\ k = 1, 2, \cdots;\ j$与$2k+1$互质)时:

$B'(a_\theta) = 0$的根为$a_\theta^1 = -\dfrac{B_2}{B_1\Big[\dfrac{1-\beta(1-\gamma)}{\gamma} + 1\Big]}$和$a_\theta^2 = -\dfrac{B_2}{B_1}$. 另外, $B_1 > 0, B_2 < 0$. 由$B_1 > 0$和$B_2 < 0$,可知$a_\theta^1 < a_\theta^2$.

若$A(0) < \lim\limits_{a_\theta \to 0} B(0)$,即$A_2 < -\eta_t$时,方程一定有正根.

若$A(0) \geqslant \lim\limits_{a_\theta \to 0} B(0)$,即$A_2 > -\eta_t$时:

a) 当$0 < -\dfrac{A_1}{2A_0} < a_\theta^2$时,若$A\Big(-\dfrac{A_1}{2A_0}\Big) \leqslant B\Big(-\dfrac{A_1}{2A_0}\Big)$,则方程一定有正根;若$A\Big(-\dfrac{A_1}{2A_0}\Big) > B\Big(-\dfrac{A_1}{2A_0}\Big)$,则方程有正根的条件为$\exists a'_\theta \in (0, a_\theta^2)$, $A(a'_\theta) \leqslant B(a'_\theta)$.

b) 当$-\dfrac{A_1}{2A_0} \geqslant a_\theta^2$时,方程有正根的条件为$A(a_\theta^2) \leqslant B(a_\theta^2)$.

iii) 当$\dfrac{1-\beta(1-\gamma)}{\gamma} = 2n+1(n = 1, 2, \cdots)$或$\dfrac{1-\beta(1-\gamma)}{\gamma} = \dfrac{j}{2k+1}(j > 2;\ j$为奇数$;\ k = 1, 2, \cdots;\ j$与$2k+1$互质)时:

a) 若$A\Big(-\dfrac{A_1}{2A_0}\Big) \leqslant B\Big(-\dfrac{A_1}{2A_0}\Big)$时,则方程一定有正解;

b) 若$A\Big(-\dfrac{A_1}{2A_0}\Big) > B\Big(-\dfrac{A_1}{2A_0}\Big)$时,则方程有正根的条件为$\exists a'_\theta > 0, A(a'_\theta) \leqslant B(a'_\theta)$.

归纳以上讨论,得到如下结论.

定理1 假设$A_2 < 0$, a_θ^*是方程(32)的正根. 若a_θ^*满足$a_\theta > 0$,则基于易腐品和不可分割耐用品为消费品的最优消费投资与寿险购买的值函数为

$$V(x, s) = \frac{1}{1-\gamma} a_v s^{-(1-\beta)(1-\gamma)} x^{1-\gamma}. \tag{45}$$

最优策略为

$$\pi(t)^* = a_\pi x(t),\ \theta(t)^* = a_\theta^* x(t), \tag{46}$$

$$c(t)^* = a_c x(t),\ \bar{p}(t)^* = a_p x(t). \tag{47}$$

其中: x_t是财富过程, a_c、a_π和a_p的表达式见式(30)、(28)和(31).

3 数值模拟

本节通过数值分析方法来分析模型,并将模型中投资者的投资行为与现实中投资者的投资行为进行对比. 假设金融市场上存在一个无风险资产和一个风险资产. 在t时刻的相关参数见表1.

表1 建立模型所用参数

变量	参数值	变量	参数值
λ	0.15	γ	0.5
β	0.1	μ	0.05
σ	0.5	r	0.03
σ_1	1	η	0.5
σ_2	3	δ	0.03
b	0.05	ρ	10
s	1		

本文从死亡率、耐用品期望收益率、耐用品价格的变化来分析投资者最优策略的变化情况,如图1～图4所示.

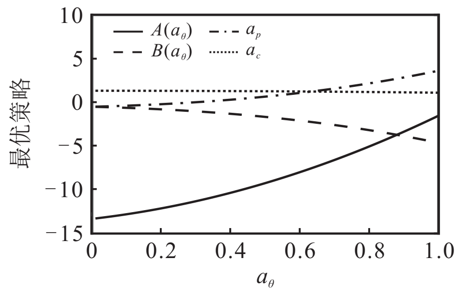

图1 参数如表1时的最优策略

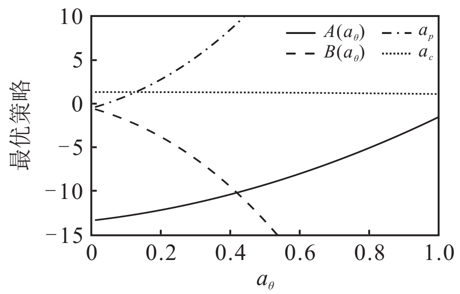

图2 $\lambda = 0.5$,其他参数值如表1时的最优策略

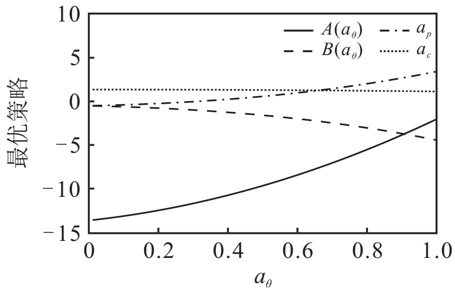

图 3　$\mu=0.3$,其他参数值如表1时的最优策略

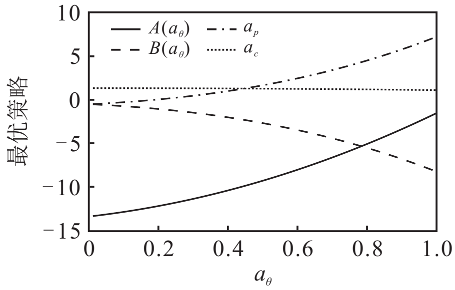

图 4　$s=2$,其他参数值如表1时的最优策略

图1～图4中,$A(a_\theta)$与$B(a_\theta)$函数图像交点横坐标为a_θ^*. 从图像可直观看出,当$a_\theta=a_\theta^*$时,$a_c>0$且$a_p>0$,满足假设条件,可得到最优策略为$\pi(t)^*=a_\pi x(t),\theta(t)^*=a_\theta^* x(t),c(t)^*=a_c x(t),\bar{p}(t)^*=a_p x(t)$.

对比图1和图2可知, 随着死亡率的上升,投资者寿险购买的占比提高,对耐用品的投资降低,这与现实中投资者的行为相符合,因为当投资者的死亡概率上升时,寿险能够提供更高的效用.

对比图1和图3可知,随着耐用品的期望收益的提高,投资耐用品的资金量提高,这与现实中投资者的行为也相吻合.

对比图1和图4可知,当耐用品的价格提高时,投资耐用品的资金占比下降,这也是现实中理性经济人的投资行为,并且符合价值规律.

综上所述,模型对于现实中投资者的投资行为具有良好的刻画,因而投资者面临寿险购买和耐用品投资时,可以采用该模型分析投资者的最优消费投资与寿险策略.

4　结　论

本文研究基于易腐品和不可分割耐用品的最优消费投资与寿险购买策略. 首先,将寿险购买引入投资者的投资组合之中,使模型的投资组合更贴近于现实投资者的投资组合;然后,将消费品进一步细划分为易腐品与不可分割耐用品,并设定投资者的投资目标为期望效用最大化;最后,通过HJB方程得到了投资者最优策略满足的方程,并进一步讨论了方程存在正根的条件. 运用Matlab软件进行数值分析,验证了模型与现实投资者投资行为的一致性.

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